Influence of dissipation on gravity waves propagating through a shear layer and on instabilities : Validity of the linear approximation

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ABSTRACT. The influence of various types of dissipation on the behaviour of gravity waves propagating in a shear layer and on shear instabilities are analysed. Among the instabilities, we study those found by Drazin (1958) in an hyperbolic tangent wind profile, and another type which arises when stratification vanishes at the critical level. The interest of this last type of instabilities consists in the fact that they are propagating waves and can appear without rigid boundary conditions. The results we found concern linear approximation, which as is usually assumed, is valid for small amplitude waves. Nevertheless, the limit of the validity has not been well defined ; this is why we have performed calculation with a fully non-linear model to determine this limit. We found that in general the linear approximation is restricted to waves whose maximum horizontal induced velocity is less than 10 % of the velocity scale of the mean flow.

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1. INTRODUCTION

The theory of gravity waves propagating through a horizontal shear layer depending on altitude has been extensively studied. Difficulties arise when the mean flow matches the horizontal phase velocity of the wave, producing a critical level.

In many studies, it is assumed that the fluid is inviscid, non rotating and that the wave amplitude is sufficiently small to use the linear approximation. In this case, the vertical structure of the wave is governed by the Taylor-Goldstein equation. In this context, Booker and Bretherton (1967) showed that transmission of the wave through a critical level depends only on the Richardson number at this level. They only considered background flow with Richardson number larger than 0.25 and found that the wave is highly damped as it propagates through the critical level. Later on, Jones (1968) found that for low values of the Richardson number at the critical level, overreflection and overtransmission can occur.

These results were confirmed using different background flows : a broken line profile (Eltayeb and McKenzie, 1975), a hyperbolic tangent profile (Van Duin and Kelder, 1982) and a continuous jet profile containing two critical levels (Teitelbaum and Kelder, 1985).

Viscosity and thermal conduction were introduced by Hazel (1967). He has shown that for values of the Richardson number larger than 0.25, a large amount of wave energy is lost near the critical level. Van Duin and Kelder (1986) proved analytically that for large values of the Reynolds number, the reflection and transmission coefficients remain almost unchanged. Teitelbaum *et al.* (1986) confirmed numerically this result, at least qualitatively.

Geller *et al.* (1975) and Fritts and Geller (1976) studied the instability in the vicinity of the critical level. They found that viscosity and heat conduction may have a strong stabilizing role. A numerical model was used by Fritts (1978, 1982) to compare the effects of viscosity, time dependence and non-linear interactions. Time dependence is found to play only a minor role in stabilizing the critical level. Non-linear effects can give rise to higher harmonics of the forcing wave, which develop large amplitudes near the critical level when viscous effects are small.

A non-linear non dissipative treatment was given by Brown and Stewartson (1980, 1982a, b). They have shown that for large values of the Richardson number, the linear model is valid up to a certain time inversely proportional to the wave amplitude.

Another very different approach to the non-linear stationary problem was given by Teitelbaum and Sidi (1979). They have shown that a contact discontinuity appears below the critical level in the absence of dissipation. As a matter of fact, non-linearities might change the results of linear models to an extent that is not well understood neither theoretically nor experimentally.

More recently, Miller and Lindzen (1988) studied the influence of viscosity and thermal conduction on the reflection of a wave interacting with a shear layer and on the instabilities which arise when a rigid lower boundary is included. In this linear approach, the authors found that dissipative processes may have a destabilizing effect as it was previously stated by Lindzen and Rambaldi (1986) for a Poiseuille flow.

In this study, we analyse the influence of various types of dissipation on the behaviour of gravity waves and instabilities. For this, we consider some cases close to those studied by Miller and Lindzen (1988) (i.e. reflection and transmission of gravity waves through a critical level and instabilities).

Among the instabilities, we have studied those found by Drazin (1958), in an hyperbolic tangent profile (thereafter called Drazin instabilities) and another type which arises when the stratification vanishes at the critical level.

Our results are obtained using linear approximation. It is usually assumed that this approximation is valid when the wave amplitude is sufficiently small. Nevertheless, the threshold is not well defined. It is the reason why in Section 2 we compare the results obtained with the linear approximation to a time-dependent non-linear model in order to test the validity of the linear approximation used in Sections 3-6. The limitations of the linear approximation concern the maximum wave amplitude and the time during which both simulations give nearly the same reflection and transmission coefficients of a gravity wave propagating through a critical level.

In Section 3, using a numerical stationary linear model, we study the influence of dissipations on the reflection and transmission of gravity waves propagating through different wind profiles.

The influence of dissipations is also studied in the problem of instabilities in Section 4. Furthermore, non-linear simulations are made to test the validity of the linear approximation.

In Section 5, we use Rayleigh friction and Newtonian cooling instead of viscosity and thermal conduction.

In Section 6, we simulate a turbulent shear layer surrounding the critical level.

2. DESCRIPTION OF THE TIME-DEPENDENT MODEL AND COMPARISON BETWEEN LINEAR AND NON-LINEAR RESULTS

a. The model

Using the hydrostatic approximation, the non-linear system of equations are written in dimensionless form. We take d, U_0, d/U_0 and $N_0^2 d^2$ as units of length, speed, time and geopotential height. d is the thickness of the shear layer, U_0 and N_0 are typical values of the background velocity and Brunt-Väisälä frequency.

We introduce a stream function ψ:

$$\psi_z = pu \qquad \psi_x = -pw \tag{1}$$

the function:

$$R = \psi_{zz} + \psi_z/h \tag{2}$$

where $p = e^{-z/h}$ and $h = H_0/d$ (H_0 is a mean scale length of the atmosphere) and the geopotential height $\varphi = \int g\, dz$.

The equations are:

$$\begin{aligned} &R_t + \psi_z R_x/p - \psi_x R_z/p - \frac{2\psi_x R}{ph} + UR_x - \\ &- \left(U_{zz} + \frac{U_z}{h}\right)\psi_x + pJ\varphi_{zx} \\ &- \left(\frac{1}{Re}\right)\left(R_{zz} + \frac{2R_z}{h} + \frac{R}{h^2} + R_{xx}\right) \\ &- \left(\frac{1}{Re}\right)_z \left(R_z + \frac{R}{h} + \psi_{zxx}\right) + a_z \psi_z + aR = 0 \end{aligned} \tag{3}$$

$$\begin{aligned} &\varphi_{zt} + \frac{\psi_z \varphi_{zx}}{p} - \frac{\psi_x \varphi_{zz}}{p} - \frac{\gamma - 1}{\gamma}\frac{\psi_x \varphi_z}{ph} + U\varphi_{zx} - \\ &- n^2 \frac{\psi_x}{p} - \frac{1}{Pr\, Re}(\varphi_{zzz} + \varphi_{zxx}) + b\varphi_z = S \end{aligned} \tag{4}$$

$U(z)$ is the normalized initial mean wind;
$n(z)$ is the normalized initial Brunt-Väisälä frequency;
J is a Richardson number $N_0^2 d^2/U_0^2$;
Re is a Reynolds number $U_0 d/\nu$;
Pr is the Prandtl number ν/K;
a is the normalized Rayleigh friction and b the normalized Newtonian cooling coefficient;
ν is the viscosity coefficient and K is the thermal conduction coefficient.

The driving wave mechanism is a heating source S located far below the shear layer. It is given by:

$$S = f(t)\, e^{-[(z-z_s)/\Delta z_s]^2}\, e^{ik(x-ct)}. \tag{5}$$

k is the horizontal wavenumber and c is the phase velocity of the fundamental mode.

$f(t)$ is a slowly varying function which is zero at the initial time and reaches a constant value after one period $2\pi/kc$ of the fundamental harmonic. It makes it possible to limit the amplitude of the transient modes which frequencies are very different from kc.

At the top and at the bottom of the field, we impose very thick sponge layers in order to prevent reflections at the boundaries and to introduce very simple boundary conditions:

$\psi = 0$ at the top and at the bottom of the field.

In the horizontal direction, we apply periodic boundary conditions.

The spatial derivatives in the x and in the z directions are calculated with centered finite differences.

As a very good resolution is required near the critical level (the gridspacing must be smaller than the viscous length $(kRe)^{-1/3}$; Hazel, 1967), we use a stretched grid in the vertical.

The use of such a gridspacing requires some precautions.

Actually, far from the critical layer, as the gridspacing becomes large, we are not able to integrate the modes which are due to the viscosity and the thermal conduction and whose vertical wavelengths are short. Furthermore, a change in the gridspacing can induce numerical reflections and waves can be trapped in the high resolution zone (i.e. the critical layer). These remarks make it necessary to compare in a few cases the results obtained when we used a stretched grid with those obtained when the gridspacing is small and constant in the whole field.

When the change in the gridspacing is smooth, it appears that the time evolution of the propagation of the gravity wave through the critical layer is close to that obtained with a constant gridspacing.

Furthermore, the structure of the perturbed field is nearly the same in both cases.

The temporal integrations of R and φ_z are performed with a predictor corrector algorithm due to Hyman (1979) and used in Lindzen and Barker (1985). The vertical integration of the ψ function is calculated by a Gaussian elimination technique.

Figure 1 represents values of the field $\psi_z(x, z, t)$ for a fixed X and for two different times. It illustrates the propagation of the gravity wave in the field described above.

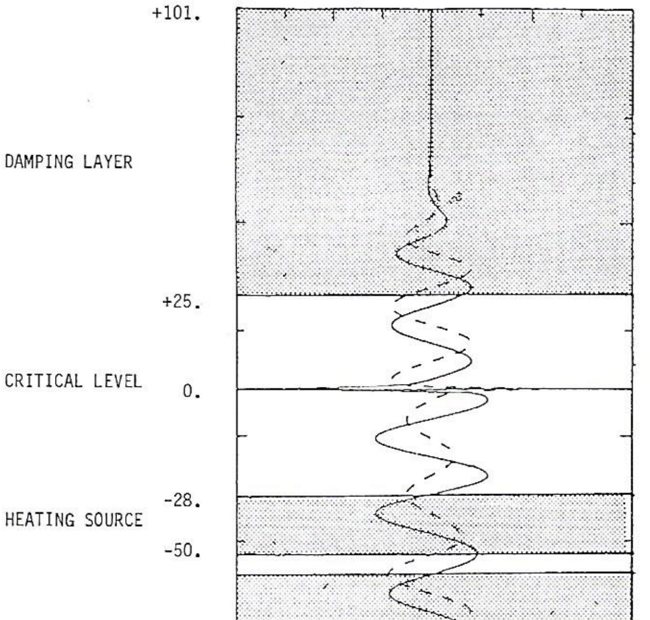

Figure 1

Vertical structure of the field used in the model and profile of the wave induced horizontal wind velocity $u\, e^{-z/2H}$ *at a given time (——) and a quarter of period later (------). The mean wind is an hyperbolic tangent profile* $U(z) = 1 + th(z)$ *while the Brunt-Väisälä frequency is constant and* $C = 1$ *(i.e. the critical level is situated at* $z = 0$*).*

b. Limitations of the linear approximation

The limitation of the linear approximation is tested for different values of the viscous parameters. In all the experiments, we impose :

$$U(z) = 1 + th(z)\ ;\quad n^2(z) = 1\ ;\quad c = 1\ ;$$
$$k = 2 \cdot 24\, 10^{-2}\ ;\quad J = 0.2 \quad \text{and} \quad h = 31.3\ .$$

We note that there exists a critical level at $z = 0$.

In order to test the linear approximation, we increase the amplitude of the wave source (i.e. the amplitude of the wave propagating toward the critical level) until an amplitude for which the differences between the values of the linear and non-linear reflection and transmission coefficients remain smaller than 5 %.

Note that, in these simulations, the initial background wind profile is free of dissipation. The action of dissipative phenomena on the initial background wind profile is the same in the linear and non-linear cases. As we are interested in the differences between both approximations, this variation is irrelevant. Physically, we can say that the initial mean background wind is maintained by a source which compensates dissipative effects. Note that the modifications of the background wind by non-linear effects are taken into account and undergo dissipation.

In Figure 2, the time evolution of the reflection and transmission coefficients in the non-linear case (Fig. 2*a*) and in the linear case (Fig. 2*b*) is represented for $Re = 4.4\, 10^4$ and $Pr = 1$. The stationary state is reached in both approximations after 7 periods of the driving force and the reflection and transmission coefficients remain the same up to at least 13 periods.

We find that the largest value of the horizontal velocity of the incident wave at which the non-linear effects can be neglected is equal to $5\, 10^{-3}$. The maximum amplitude of the horizontal wind induced by the wave in the whole field is equal to $2.5\, 10^{-2}$. It corresponds to the simulation represented in Figure 2*a*. In dimensional form, if we consider realistic scales : $U_0 = 10$ m s^{-1}, $N_0^2 = 4\, 10^{-4}$ s^{-2} ; it means that for a viscosity equal to 0.05 m^2 s^{-1}, the horizontal velocity of the incident wave near the source is 0.05 m s^{-1}, while the maximum amplitude of the horizontal wind induced by the wave near the critical level is 0.25 m s^{-1}. The above results show that for the chosen values of the viscosity, the linear solution is limited to rather small amplitudes.

The results displayed in Figure 2*c* correspond to a similar non-linear simulation when the amplitude of the incident wave is three time that of the preceding case. A stationary state is reached while the reflection coefficient obtained is larger than in the linear case. This difference is due to the fact that during the transient period, the background conditions slightly change and the Richardson number at the critical level decreases to a value equal to 0.16.

For other experiments with higher values of the wave amplitude and the same viscosity, the non-linear simulations never reach a stationary state.

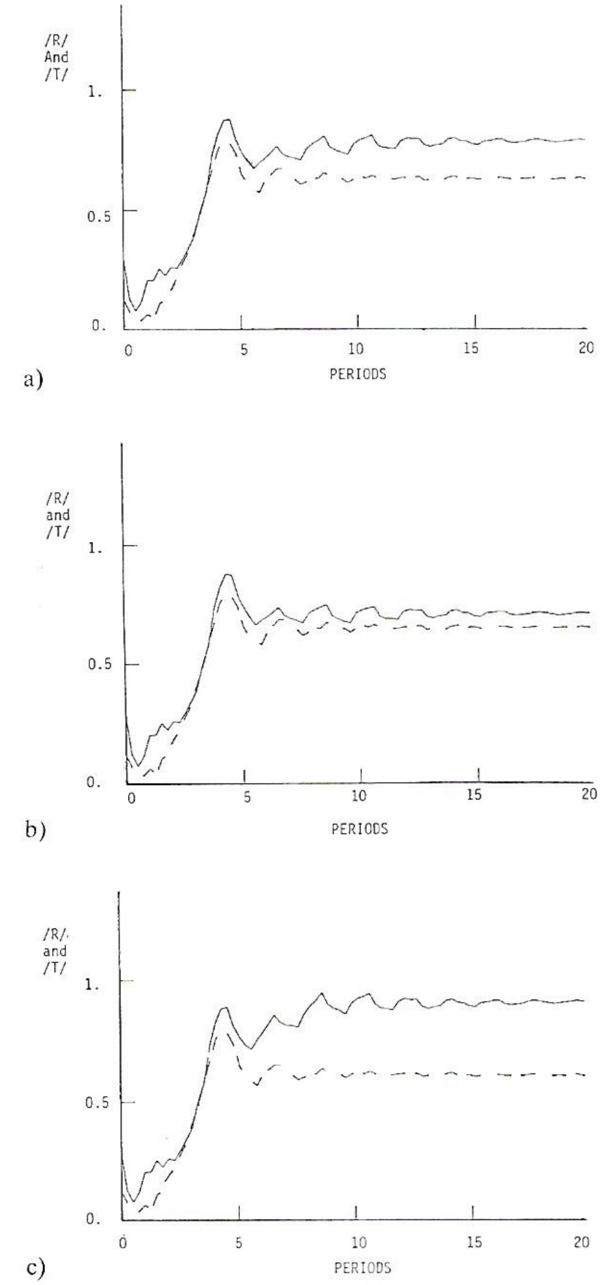

Figure 2

Temporal evolution of the reflection (——) and transmission (------) coefficients. $U(z) = 1 + th(z)$; $n^2(z) = 1$; $c = 1$; $k = 2.24\ 10^{-2}$; $J = 0.2$; $Pr = 1$; $h = 31.2$ *and* $Re = 4.4\ 10^4$. (a) *Non-linear case ; the amplitude of the incident wave is* 5.10^{-3} ; (b) *Linear case ;* (c) *Non-linear case ; the amplitude of the incident wave is* $1.5\ 10^{-2}$.

We repeat the same experiments for $Re = 4.4\ 10^3$ and $Pr = 1$. We find that the linear approximation remains valid up to a value of the incident wave equal to $3\ 10^{-2}$ ($0.3\ \mathrm{m\ s^{-1}}$) ; the maximum horizontal wind amplitude induced by the wave is equal to 0.11 ($1.1\ \mathrm{m\ s^{-1}}$; Fig. 3).

The validity of the linear approximation for higher values of the wave amplitude and more dissipation than in the preceding case means that the viscosity and the thermal conduction act against non-linearities.

General dissipation inhibits the growth of secondary modes which appear due to the non-linear cascade. Furthermore, dissipation does increase the thickness of the layer in which the momentum and thermal exchanges between the wave and the mean flow take place. The last effect means that for the same wave amplitude and for larger viscosity, the modification of the mean flow is less important than in a thicker layer. As a consequence, the reflection and the transmission of the wave are less modified.

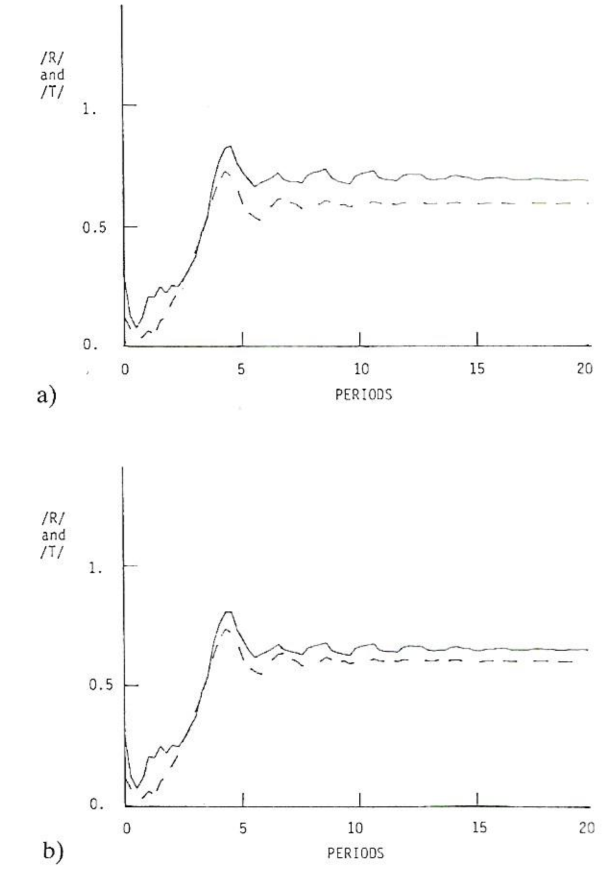

Figure 3

Same as Figure 2 except that $Re = 4.4\ 10^3$. (a) *Non-linear case ; the amplitude of the incident wave is* $3 \cdot 10^{-2}$; (b) *Linear case.*

Such a mechanism is subjected to limitations, since we cannot indefinitely increase the amplitude of the perturbed field by increasing the viscosity and the thermal conduction to keep the system stationary. In other experiments with higher viscosity ($Re = 4.4\ 10^2$ and $Pr = 1$), the maximum amplitude of the incident wave for which the linear approximation is valid remains very close to that obtained when $Re = 4.4\ 10^3$. Actually, we can remark that, for such viscosity, the deposition of momentum and temperature on the mean flow are significant in the whole field and are not closely concentrated around the critical level (the viscous length $(kRe)^{-1/3} \simeq 0.3$). From this point, any increase in the viscosity will increase the momentum and thermal deposition far under the critical level and the amplitude of the wave for which the linear approximation remains valid is now limited by this process.

These results permit us to give some general quantitative appreciation of the linearisations frequently adopted to treat this kind of problems. One can say that the linear approximation is not valid if the amplitude of the incident wave exceeds $3 \cdot 10^{-2}$ and if the amplitude of the largest horizontal wind induced by the wave near the critical level exceeds 0.1

($0.3\ \mathrm{m\,s^{-1}}$ and $1\ \mathrm{m\,s^{-1}}$ in dimensional form if $U_0 = 10\ \mathrm{m\,s^{-1}}$). These values are of course approximative because they are deduced from numerical experiments and for particular values of the parameters, but we think that they give an idea about the limitations of the linear approximation.

Our results differ from those obtained by Brown and Stewartson (1980, 1982a, b) who found that a stationary state cannot persist for a very long time because secondary modes generated around the critical level increase in time and modify the reflection and transmission coefficients. This is not surprising since Brown and Stewartson considered the inviscid case.

Furthermore, we observe that viscosity and thermal conduction play the same role in preventing non-linear effects. In fact, the maximum amplitude for which the linear approximation is valid when $Pr \neq 1$ remains the same if we permute the values of viscosity and thermal conduction.

Nevertheless, if the wave amplitude is such that the non-linear effects become important, the role of viscosity and thermal conduction differs. Figure 4 displays the results concerning the time evolution of the reflection and transmission coefficient when $Re = 2.2\ 10^4$ while $Pr = 5$ in the non-linear (Fig. 4*a*) and in the linear cases (Fig. 4*b*). Until five wave periods, the linear and the non-linear results are the same. From this point, linear results remain stationary whereas non-linear ones show increasing values of the reflection and transmission coefficients.

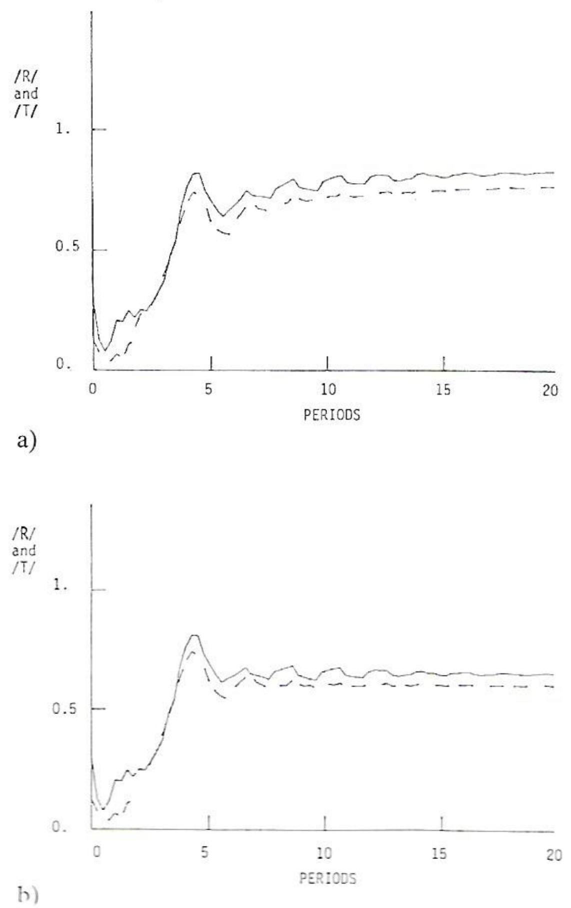

Figure 4

Same as Figure 2 except that $Re = 4.4\ 10^3$ *and* $PrRe = 2.2\ 10^4$.
(a) *Non-linear case ; the amplitude of the incident wave is* $1.5\ 10^{-2}$;
(b) *Linear case.*

For $Pr = 0.2$, $Re = 4.4\ 10^3$, the non-linear results (Fig. 5*a*) are stationary, but the wave transmission is smaller than in the linear simulation (Fig. 5*b*).

The temporal variation of the reflection and transmission coefficients when $Pr = 5$ can be explained by the modification of the background wind profile : viscosity is more important than thermal conduction in the wave-mean flow momentum exchange.

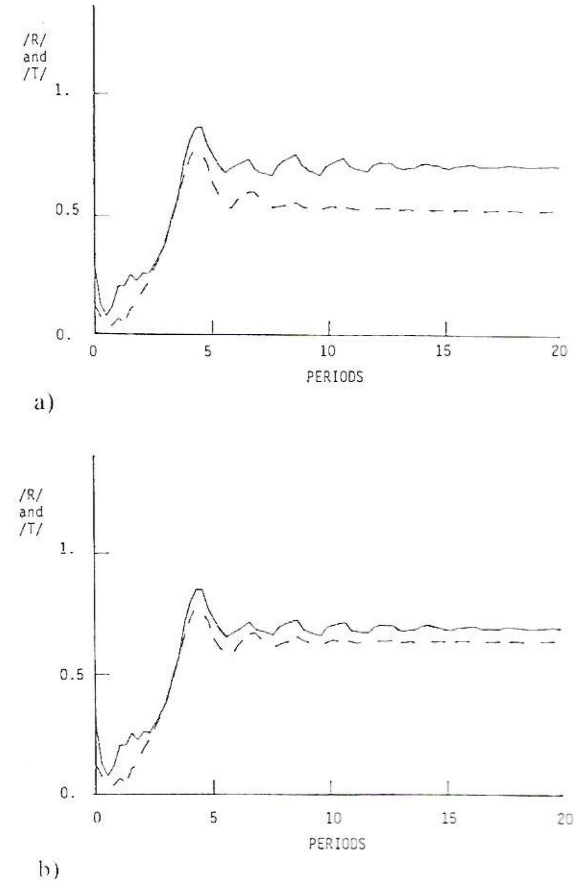

Figure 5

Same as Figure 2 except that $Re = 2.2\ 10^4$ *and* $PrRe = 4.4\ 10^3$.
(a) *Non-linear case ; the amplitude of the incident wave is* $1.5\ 10^{-2}$;
(b) *Linear case.*

3. EFFECTS OF DISSIPATION ON A LINEAR GRAVITY WAVE PROPAGATING THROUGH A SHEAR LAYER

a. Stationary model

The following results regard the influence of molecular viscosity and thermal conduction on the reflection and the transmission of a gravity wave interacting with a critical level. The approach is linear and we use a stationary numerical model. Nevertheless, some of the results have been verified with the time dependent model in its linearized version. The principle of the stationary model is similar to the one used by Lindzen and Rambaldi (1986), for the viscous Poiseuille flow. We develop an algorithm which calculates directly the stationary modes of the viscous problem :

$F(z)\, e^{ik(x-ct)}$ in the hydrostatic approximation. The horizontal wave number k and the phase velocity c are real. All the variables are normalized as for the time-dependent model.

We got to solve a system of three second order equations :

$$AF_{zz} + BF_z + CF = S \quad (6)$$

where :

$$F(z) = \begin{pmatrix} R(z) \\ \varphi_z(z) \\ \psi(z) \end{pmatrix}. \quad (7)$$

The functions R, φ_z and ψ are the same as those of the preceding section

$$A(z) = \begin{pmatrix} i/kRe & 0 & 0 \\ 0 & i/kPrRe & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (8a)$$

$$B(z) = \begin{pmatrix} \frac{2i}{kReh} + \left(\frac{i}{kRe}\right)_z & 0 & -i\left(\frac{a}{k}\right)_z \\ 0 & 0 & 0 \\ 0 & 0 & 1/h \end{pmatrix} \quad (8b)$$

$$C(z) = \begin{pmatrix} U - c - i\frac{a}{k} - i\frac{k}{Re} + \frac{i}{kReh^2} + \left(\frac{i}{kReh}\right)_z & Jp & -U_{zz} - \frac{U_z}{h} \\ 0 & U - c - i\frac{b}{k} - \frac{ik}{PrRe} & -\frac{n^2}{p} \\ -1 & 0 & 0 \end{pmatrix} \quad (8c)$$

All the parameters H, Re, Pr, h, a, b, as well as $U(z)$, $n(z)$, c and k are identical to those introduced in the non-stationary model

$$S = \begin{pmatrix} 0 \\ J_h(z) \\ 0 \end{pmatrix}. \quad (9)$$

$J_h(z)$ is a heating source located sufficiently far below the critical level and which generates an incident wave propagating towards the critical level.

In the preceding equations, we note that all but one of the viscous terms are inversely proportional to kRe. In fact, the influence of the viscosity and of the thermal conduction depends only on this parameter and on the Prandtl number. This is not surprising since $(1/kRe)^{1/3}$ is the normalized viscous length introduced by Hazel (1967).

In order to avoid the influence of the boundaries, we introduce very thick sponge layers to inhibit reflections at the top and at the bottom of the field. The choice of the boundary conditions doesn't affect the solution because the wave is completely attenuated there. We impose then $\psi = 0$ at the top and at the bottom of the field (9).

The system is integrated using a Gaussian elimination technique.

The reflection coefficient is computed at an height z_m located above the source but sufficiently far below the critical layer, such that

$$\left(\frac{dU}{dz}\right)_{z=z_m} = 0.$$

At this level, we may separate the solution in 6 modes for which vertical wavenumbers ℓ_i are given by the viscous dispersion relation :

$$\frac{1}{PrRe^2}\frac{\ell^6}{k^2} + \frac{i}{PrRe^2}\frac{\ell^5}{k^2 h} - ic\frac{Pr+1}{PrRe}\frac{\ell^4}{k} + c\frac{Pr+1}{PrRe}\frac{\ell^3}{kh} - c^2\ell^2 - ic^2\frac{\ell}{h} + J = 0. \quad (10)$$

As in the Lindzen and Rambaldi (1986) formulation of the Poiseuille problem, we can identify two modes for which the vertical wavenumbers are close to the wave numbers of the inviscid problem. The whole solution is given by the superposition of these two inviscid modes and four viscous modes. The stream function may be written as :

$$\psi(z_m) = Up\, e^{i\ell_1 z_m} + Do\, e^{i\ell_2 z_m} + A_{PrRe}\, e^{i\ell_3 z_m} + B_{PrRe}\, e^{i\ell_4 z_m} + C_{PrRe}\, e^{i\ell_5 z_m} + D_{PrRe}\, e^{i\ell_3 z_m} \quad (11)$$

Up is the amplitude of the incident « inviscid » wave. Do is the amplitude of the reflected « inviscid » wave. A_{PrRe}, B_{PrRe}, C_{PrRe}, D_{PrRe} are the amplitudes of the 4 viscous modes.

The viscous modes are important at the shear layer and near the source, but they are very strongly absorbed as they propagate away. At the height z_m where the reflection coefficient is calculated :

$$\psi(z_m) \simeq Up\, e^{i\ell_1 z_m} + Do\, e^{i\ell_2 z_m} \quad (12)$$

within a very good approximation. The reflection coefficient is :

$$R = \frac{|Do|}{|Up|}. \quad (13)$$

The same mode separation is made at an height z_T located far above the critical level. This makes it possible to choose the sponge layers parameters such that the downward propagating modes (i.e. the modes that might be reflected from the upper boundary) are negligible in comparison with the upward inviscid propagating mode. We can write there :

$$\psi(z_T) \simeq Up' \, e^{i\ell_1 z_T} \tag{14}$$

and the transmission coefficient is :

$$T = \frac{|Up'|}{|Up|} . \tag{15}$$

After (12), (13), (14) and (15), we note that the reflection and transmission coefficients defined there are independent on the height where they are calculated.

We consider that the results obtained by this way take only into account the influence of the dissipation at the critical level.

b. Results

In order to test the results of Van Duin and Kelder (1986), realistic values of the mean profiles and of the viscous parameters are considered. These authors found that for large Reynolds number, the reflection and transmission of the wave are the same as in the inviscid case. In figure 6, we consider realistic scale length $U_0 = 10 \text{ m s}^{-1}$ and $N_0^2 = 4\,10^{-4} \text{ s}^{-2}$ in order to test the influence of viscosities with values consistent with those find in the stratosphere ($\nu = 0.01 \text{ m}^2 \text{ s}^{-1}$) and in the mesosphere ($\nu = 0.5 \text{ m}^2 \text{ s}^{-1}$).

The dimensional wavenumber of the wave is $k = 10^{-4} \text{ m}^{-1}$.

We compare these results with those of the inviscid case found by Van Duin and Kelder (1982) for the hyperbolic tangent wind profile. For $\nu = 0.01 \text{ m}^2 \text{ s}^{-1}$, the wave is reflected like in the inviscid case. For $\nu = 0.5 \text{ m}^2 \text{ s}^{-1}$, the reflection coefficient is lowered. Figure 7 displays the same results for the jet profile

$$\begin{pmatrix} U(z) = (1 + z^2)^{-1} \\ n^2(z) = 1 \end{pmatrix}$$

with two critical levels compared to the inviscid case found by Teitelbaum and Kelder (1985). Note that similar results are obtained for the transmission coefficient.

Also note that low values of the viscosity do not prevent the existence of overreflection.

Reflection coefficients, expressed as a function of the nondimensional viscous parameters are reported in Table 1, for a given Richardson number $J = 0.15$. The mean wind is a hyperbolic tangent profile. For $kRe > 1000$, variations of the reflection coefficient as a function of the viscosity are small and are independent of the thermal conduction (i.e. the Prandtl number). For $kRe < 1000$, the effect of the viscosity becomes important while the influence of the thermal conduction is always small.

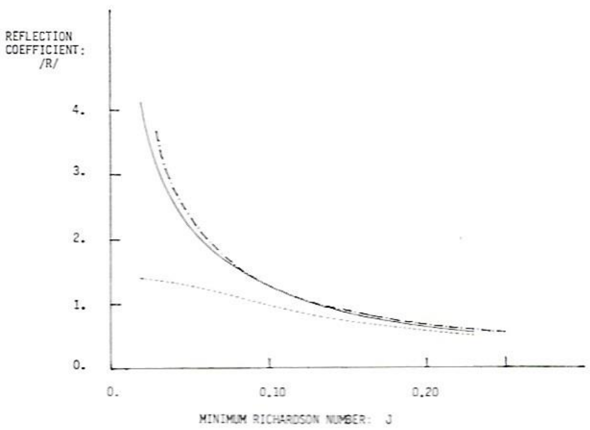

Figure 6

Variation of the reflection coefficient as a function of the minimum Richardson number. $U(z) = 1 + th(z)$; $n^2(z) = 1$; $c = 1$ *and* $Pr = 1$. *While the dimensional parameters we consider are :* $U_0 = 10 \text{ m s}^{-1}$; $h_0 = 7000$ m ; $k = 10^{-4} \text{ m}^{-1}$. (-····-) $\nu = 0 \text{ m}^2 \text{ s}^{-1}$; (——) $\nu = 0.01 \text{ m}^2 \text{ s}^{-1}$; (------) $\nu = 0.5 \text{ m}^2 \text{ s}^{-1}$.

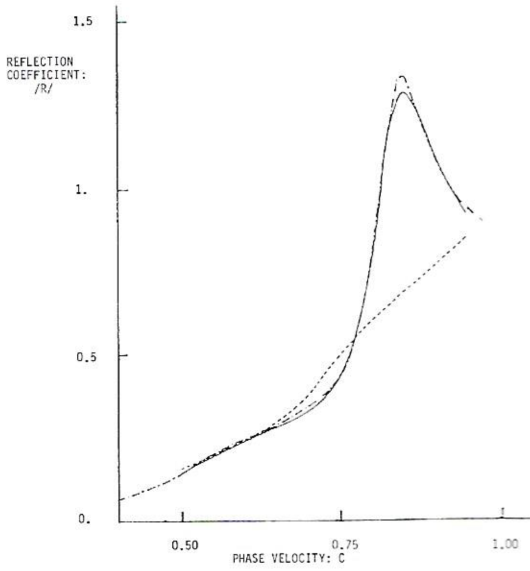

Figure 7

Variation of the reflection coefficient as a function of the positions c of the critical level. $U(z) = (1 + z^2)^{-1}$; $n^2(z) = 1$; $J = 0.1$ *and* $Pr = 1$, $h_0 = 7000$. (-····-) $\nu = 0$; (——) $\nu = 0.01 \text{ m}^2 \text{ s}^{-1}$; (------) $\nu = 0.5 \text{ m}^2 \text{ s}^{-1}$.

Table 1

Reflection coefficient as a function of the Prandtl number and kRe. $J = 0.15$, $h = 36.15$, $c = 1$, $U(z) = 1 + th(z)$, $n^2(z) = 1$.

	kRe						
Pr	1862.0	745.0	186.0	93.1'	62.1	46.0	37.25
10.0	0.875	0.866	0.822	0.776	0.738	0.707	0.680
8.0	0.875	0.866	0.822	0.776	0.738	0.707	0.680
6.0	0.875	0.866	0.822	0.776	0.738	0.707	0.680
4.0	0.875	0.866	0.822	0.776	0.738	0.706	0.679
2.0	0.875	0.866	0.822	0.775	0.737	0.706	0.679
1.0	0.875	0.866	0.822	0.775	0.736	0.704	0.677
0.5	0.875	0.866	0.821	0.774	0.734	0.701	0.673
0.25	0.875	0.866	0.820	0.771	0.731	0.696	0.667
0.166	0.875	0.866	0.820	0.769	0.727	0.692	0.662
0.125	0.875	0.865	0.819	0.767	0.725	0.688	0.657
0.1	0.875	0.865	0.817	0.765	0.722	0.685	0.653

4. INFLUENCE OF DISSIPATION ON INSTABILITIES

a. Numerical procedures

In this Section, we consider instabilities existing without rigid boundaries. Among the instabilities, we shall consider those which, in the limit of vanishing growth rate, are evanescent and those which, in this limit, are outgoing waves, emitted spontaneously by the shear layer. As in most works concerning instabilities, we use the Boussinesq approximation.

As in Section 2, we pay attention to the limitations of the linear approximation. We adapt the non-linear non-stationary model to the Boussinesq approximation. Then, we compare non-linear simulations to linear ones.

The results concerning the influence of the dissipations in the linear approach are obtained with a stationary model very similar to that introduced in Section 3.

In fact, the equations of our stationary model can be obtained from the hydrostatic one taking $h \to \infty$ and adding to the matrix coefficient C_{33} the value $-k^2$.

Now $R = \psi_{zz} - k^2 \psi$ and $\varphi_2 = \theta'/\bar{\theta}$ where θ is the potential temperature.

The stationary numerical model required always a wave source. The use of such a model in stability problems deserves some comments. As in the case of wave reflection and transmission, we separate the solution between the wave source and the critical level in 6 modes. Above the shear layer, we perform the same separation of modes. In order to find the growth rate of the instabilities, we change the imaginary part of the phase velocity c_i until the upward « incident » inviscid mode, located between the source and the critical layer, becomes negligible compared to the downward one (the ratio between these two modes is less than 10^{-4}). In this last calculation, we don't take into account the 4 viscous modes : at the height of the separation between modes, their amplitudes are negligibly small compared to the amplitudes of the two inviscid modes. Although these viscous modes can be important near the source and near the critical level, they are very strongly absorbed when they propagate away.

This simple method makes it possible to solve numerically linear instability problems with dissipation without presuming the form of the solutions at the boundaries. Furthermore, the method can be used for propagating waves instabilities as well as for evanescent ones.

In a few experiments, we confirm the growth rate found by this trial and error method with the help of the time dependent model (adapted to the Boussinesq approximation). In this case, the driven force is a perturbation introduced around the critical level during a short time. It is possible to measure the growth rate when it becomes constant as it is expected in the linear theory.

b. Drazin instabilities

In Figure 8, we display the time development of such a growing disturbance for $J = 0.23$, $Re = 2.4\,10^4$, $Pr = 1$, while $n^2(z) = 1$, $u(z) = 1 + th(z)$ and $k^2 = 0.5$ in the linear case (Fig. 8*b*) and in the non-linear case (Fig. 8*a*).

The figure shows that the non-linear solution is consistent with the linear one at the beginning. After 20 periods, as the maximum amplitude of the perturbations becomes larger than $3\,10^{-2}$, the non-linear and the linear simulations begin to differ. It indicates that the linear approximation remains valid up to a rather small amplitude.

Considering the linear problem, the influence of viscosity on the Drazin neutral curves has been calculated for values of kRe as low as 10^3. The neutral curve is lowered in a very small amount in agreement with the work of Maslowe and Thomson (1971) using an hyperbolic tangent wind but with the particular density variation proposed by Holmboe.

The maximum Richardson number for which a neutral solution exists, decreases linearly with $(kRe)^{-1}$ and is independent of the Prandtl number as it was analytically found by Churilov and Shukhman (1987).

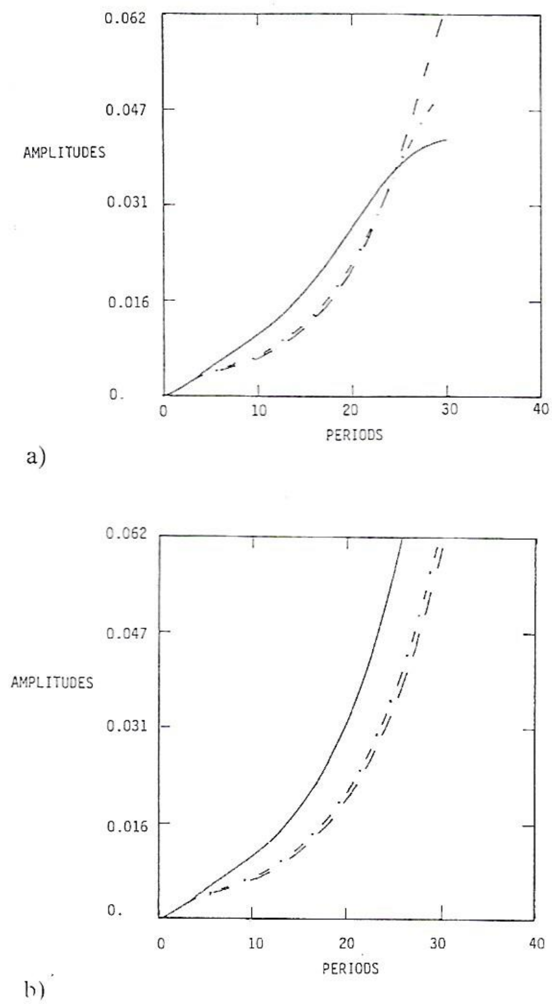

Figure 8

Time growth of a Kelvin-Helmholtz instability at different heights : (——) z_c ; (------) $z_c + 0.1$ *and* (- · · · -) $z_c - 0.1$. $U(z) = 1 + th(z)$; $n^2(z) = 1$; $c_r = 1$; $k^2 = 0.5$; $J = 0.23$; $Re = 2.4\,10^4$ *and* $Pr = 1$. (a) *Non-linear case ;* (b) *Linear case.*

For values of the kRe greater or equal to 10^3, we find :

$$J = \frac{1}{4} - 2.47/kRe\,. \quad (16)$$

This result quantitatively differs from that obtained by Churilov and Schukhman :

$$J = \frac{1}{4} - \frac{5}{4}/kRe\,. \quad (17)$$

Furthermore, we find that the growth rates of the unstable modes weakly decrease with the viscosity and are not affected by the Prandtl number.

c. Propagating waves instability

The stabilizing effect of dissipation found in the study of Drazin instabilities is not a general result. Actually, it strongly depends on the background conditions. Recently, Miller and Lindzen (1988) have shown that dissipation can give rise to instabilities in a flow with a rigid lower boundary even if the minimum Richardson number is greater than 0.25. Here, we shall consider the case when the Brunt-Väisälä frequency vanishes at the critical level and there are no reflections at the boundaries. Physically, the Brunt-Väisälä frequency can vanish at the critical level if a turbulent layer has been developed at this place. It is well known that turbulent mixing can annul the stratification. In this case, dissipation can give rise to unstable modes which don't exist in the inviscid case.

Inviscid study

Let us suppose that the Brunt-Väisälä frequency varies as a given even power of the shifted wave frequency. If the power is two, there is always resonant overreflection.

Using Boussinesq approximation in dimensionless form, the mean wind is :

$$U = 1 + th(z)\,.$$

The mean Brunt-Väisälä frequency is : $n^2(z) = th^2(z)$ and the new variable $s = th(z)$ is introduced.

For a phase velocity of the wave $c = 1$, the Taylor-Goldstein equation writes :

$$\frac{d^2w}{ds^2} + \frac{2s}{s^2-1}\frac{dw}{ds} + \left[\frac{J}{(s^2-1)^2} - \frac{2}{(s^2-1)} - \frac{k^2}{(s^2-1)^2}\right] w = 0 \quad (18)$$

where $J = N_0^2 d^2/U_0^2$; N_0^2 is representative of the Brunt-Väisälä frequency far from the critical level and J will be called the pseudo-Richardson number since the real one always vanishes at the critical level.

The equation (18) has 3 regular singular points at $s = \pm 1$ and ∞. The exponent pairs are :

$$s = +1 \begin{cases} \alpha_1 = i\,(J-k^2)^{1/2}/2 \\ \alpha_2 = -\,i\,(J-k^2)^{1/2}/2 \end{cases} \quad (19a)$$

$$s = -1 \begin{cases} \gamma_1 = \alpha_1 \\ \gamma_2 = \alpha_2 \end{cases} \quad (19b)$$

$$s = \infty \begin{cases} \beta_1 = 2 \\ \beta_2 = -1\,. \end{cases} \quad (19c)$$

Equation (18) can be transformed in the hypergeometric equation (Olver) by considering new variables :

$$t = \frac{1-s}{2} \quad \text{and} \quad w = t^{\alpha_1}(1-t)^{\gamma_1}\,W\,. \quad (20)$$

The equation for W becomes :

$$t(1-t)\frac{d^2}{dt^2}W + (e - (a+b+1) - t)\frac{d}{dt}W - abW = 0 \quad (21)$$

where :

$$a = \alpha_1 + \beta_1 + \gamma_1 = 2 + i\,(J-k^2)^{1/2} \quad (22a)$$

$$b = \alpha_1 + \beta_2 + \gamma_1 = -1 + i\,(J-k^2)^{1/2} \quad (22b)$$

$$e = 1 + \alpha_1 - \alpha_2 = 1 + i\,(J-k^2)^{1/2}\,. \quad (22c)$$

Equation (21) is the hypergeometric equation and a solution can be written (Olver) :

$$F(a, b\,; e\,; t) \quad \text{with} \quad F(a, b\,; e\,; 0) = 1\,. \quad (23)$$

In the upper half plane, the asymptotic form for $z \to +\infty$ (i.e. $s \to +1$ and $t \to 0$) of W is :

$$w = t^{\alpha_1}(1-t)^{\gamma_1} \quad (24)$$

i.e.

$$w = e^{-i\ell z} \quad \text{where} \quad \ell = (J-k^2)^{1/2}\,. \quad (25)$$

The solution (25) corresponds to an outgoing wave in the upper half plane.

In the lower half plane, the asymptotic form corresponds to $s \to -1$, or $t \to 1$, that is to say $z \to -\infty$.

We can use the relation (Olver : 10.12, Chap. 8) :

$$F(a, b\,; e\,; t) = \frac{\Gamma(e)\,\Gamma(e-a-b)}{\Gamma(e-a)\,\Gamma(e-b)}F(a, b\,; 1+a+b-e\,; 1-t) + (1-t)^{e-a-b} \times \frac{\Gamma(e)\,\Gamma(a+b-e)}{\Gamma(a)\,\Gamma(b)}F(e-a, e-b\,; 1+e-a-b\,; 1-t) \quad (26)$$

and for $z \to -\infty$:

$$w \simeq e^{i\ell z}\left\{\frac{\Gamma(e)\,\Gamma(e-a-b)}{\Gamma(e-a)\,\Gamma(e-b)} + e^{-2i\ell z}\frac{\Gamma(e)\,\Gamma(a+b-e)}{\Gamma(a)\,\Gamma(b)}\right\}. \quad (27)$$

Hence, the first term in (26) corresponds to an incident wave and the second to a reflected wave.

Furthermore $e - a = -1$; then $\Gamma(e-a) = \infty$.

The first term in (26) vanishes and with the property of the gamma function :

$$\left| \frac{\Gamma(e)\,\Gamma(a+b-e)}{\Gamma(a)\,\Gamma(b)} \right| = 1 \tag{28}$$

there is resonant overreflection (independently of the values of the wavenumber and of the pseudo Richardson number).

If the Brunt-Väisälä frequency varies as s^4, the exponent pair for the singularity located at ∞ in the Taylor-Goldstein equation is :

$$\beta_1' = \frac{1}{2} + \left(\frac{9}{4} - J\right)^{1/2} \tag{29a}$$

$$\beta_2' = \frac{1}{2} - \left(\frac{9}{4} - J\right)^{1/2}. \tag{29b}$$

In this case, resonant overreflection exists only for $J = 2$. Numerically, we find unstable modes for $J < 2$.

Here, we can remark that the instabilities we find are very different from those found by Drazin in the same wind profile and when the Brunt-Väisälä frequency is constant. In fact, the neutral curve found by Drazin (1958) corresponds to non-propagating waves with frequency greater than the Brunt-Väisälä frequency.

In the case we study, the neutral modes correspond to propagating waves.

Furthermore, the vertical variation of the Brunt-Väisälä frequency we use is very special. Nevertheless, we have verified numerically that for a Brunt-Väisälä frequency varying as a fractional power of the wave shifted frequency, similar results are obtained.

Influence of dissipation

The results obtained considering dissipations for the case $n^2(z) = s^4$ are very surprising. The value of J for which there is resonant overreflection increases with increasing dissipation (for $Pr = 1$) up to a maximum and then decreases.

In Figure 9, we show the variation of this pseudo-Richardson number at which resonant overreflection arises as a function of the viscous parameters kRe and for different values of the Prandtl number. These curves represent the locus of resonant overreflection in the plan (J, kRe) and can be considered as the neutral curves for this problem since they separated stable and unstable regions.

Considering the neutral curve ($Pr = 1$), dissipation tends to permit instabilities for a pseudo-Richardson number which is greater than that obtained in the inviscid case.

In addition, the validity of the linear approximation has been tested for this type of instability.

Figure 10 shows the time variation of the amplitude of the instability at two different heights in the non-linear (Fig. 10*a*) and in the linear case (Fig. 10*b*).

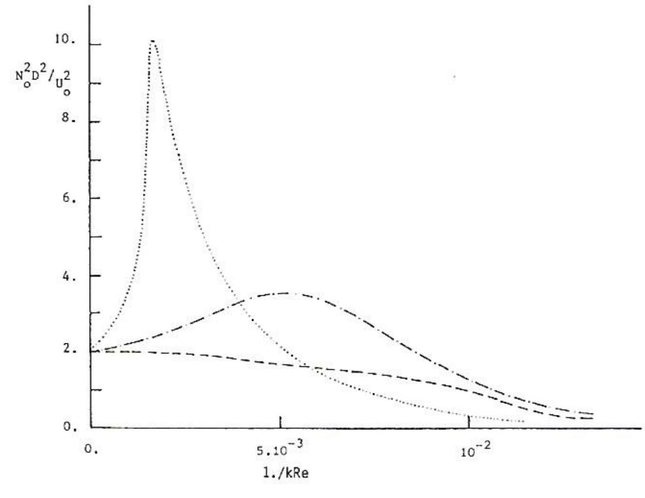

Figure 9

Variation of the pseudo-Richardson number at which resonant overreflection arises as a function of $(kRe)^{-1}$. $U(z): 1 + th(z)$; $n^2(z) = th^4(z)$. (......) $Pr = 0.33$; (- - - -) $Pr = 1$; (------) $Pr = 3$.

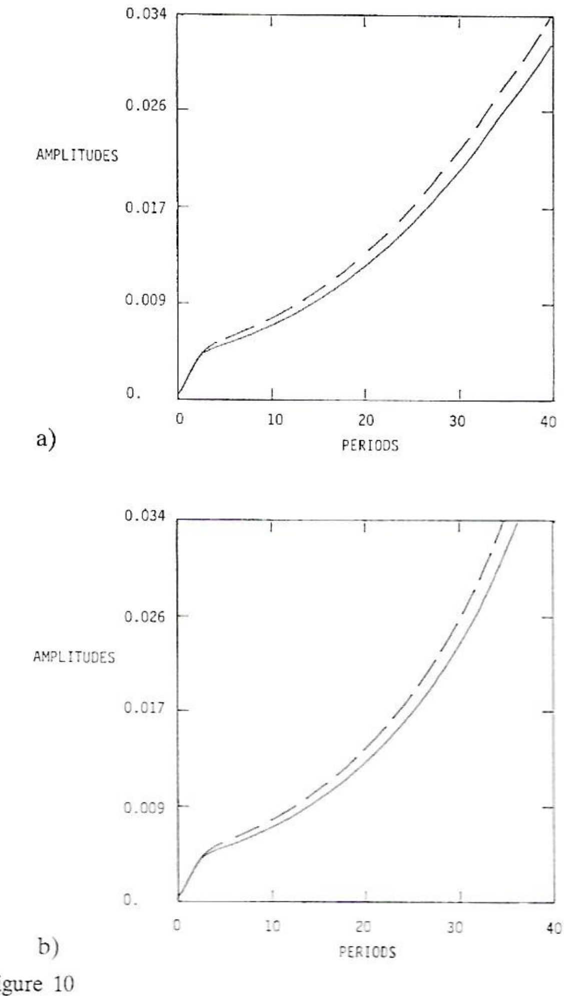

Figure 10

Time growth of a propagating wave instability at different heights : (------) z_c *and* (——) $z_c + 1$. $U(z) = 1 + th(z)$; $n^2(z) = th^4(z)$; $c_r = 1$; $k = 6.7\,10^{-2}$; $Re = 2.2\,10^4$ *and* $Pr = 1$. (a) *Non-linear case ;* (b) *Linear case.*

In this example, $J = 1.8$; $Re = 2.2\,10^4$; $Pr = 1$ and $c_r = 1$. c_r is the real part of the wave phase velocity.

The linear approximation remains valid to describe the onset of this type of instabilities as long as the amplitude is sufficiently small. Figure 11 shows the

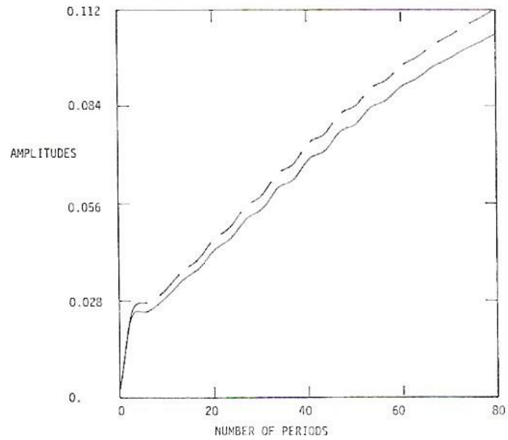

Figure 11
Similar to Figure 10a, except that the initial perturbation is larger and that the time integration is performed during 80 periods instead of 40.

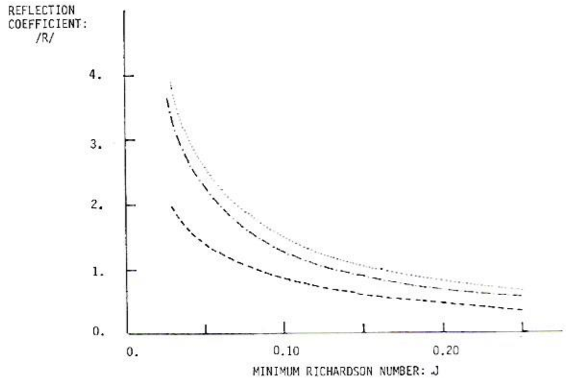

Figure 12
Variation of the reflection coefficient as a function of the minimum Richardson number. $U(z) = 1 + th(z)$; $n^2(z) = 1$; $c = 1$; $h = 31.3$ *and* $k = 2.24\ 10^{-2}$. (......) $a = 0$, $b/k = 0.1$; (- - - -) $a = 0$, $b = 0$; (------) $a/k = 0.1$, $b = 0$.

time development of the same instability during a longer time and in the non-linear case. The time growth is linear instead of being exponential. Nevertheless, we can note that non-linearities don't prevent the wave from reaching significant values of the wind speed.

5. INFLUENCE OF THE NEWTONIAN COOLING AND OF THE RAYLEIGH FRICTION

Frequently, the dissipative processes are modelled by Rayleigh friction and Newtonian cooling. In order to test the accuracy of these parameterisations, we examine the influence of Newtonian cooling and Rayleigh friction on the reflection of a linear gravity wave propagating through a critical level and on the Drazin stability problem.

Furthermore, the use of the Newtonian cooling is meaningful because it approximates the atmospheric infrared cooling and because some physical insights into the phenomena induced by dissipation at the critical level are possible.

In Figure 12, we compare the variation of the reflection coefficient of a gravity wave incident on a critical level when there is Newtonian cooling or Rayleigh friction. The background wind has an hyperbolic tangent profile ; the Brunt-Väisälä frequency is constant and we consider the hydrostatic approximation. Figure 12 shows that the Newtonian cooling tends to increase the reflection (the result is similar for the transmission). Furthermore, for sufficiently large values of the Newtonian cooling, overreflection can occur for flows having a minimum Richardson number larger than 0.25. As overreflection is linked to instabilities (Rosenthal and Lindzen, 1983a, b), we can presume that Newtonian cooling can destabilize the flow.

To confirm this assumption, we solve numerically the Drazin stability problem in presence of Newtonian cooling. This is academical because, in the atmosphere, Newtonian cooling is not sufficiently strong to influence the short period modes appearing in this problem. Nevertheless (Fig. 13), the neutral stability

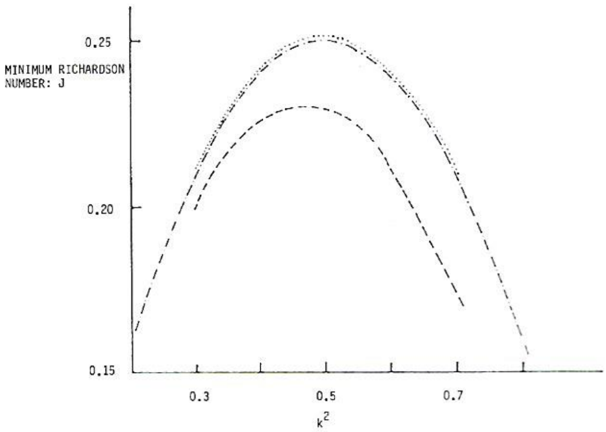

Figure 13
Neutral curve corresponding to the Drazin profile: $U(z) = 1 + th(z)$; $n^2(z) = 1$. (......) $b/k = 0.1$, $a = 0$; (- - - -) $b = 0$, $a = 0$; (------) $b = 0$, $a/k = 0.02$.

curve tends to move up when there is Newtonian cooling and there are unstable modes when the minimum Richardson number is everywhere greater than 0.25.

The importance of dissipative processes for the stability of stratified shear flows has been recently emphasized by Miller and Lindzen (1988).

The destabilizing influence of the thermal dissipation is known since Jones (1977) noted that « the radiative diffusion will weaken the stabilizing effect of the buoyancy force ».

Actually, the buoyancy force stabilizes the stratified flows because it exerts a backward force on each particle displaced from its equilibrium position. When the particle is displaced, the buoyancy force is closely related to the temperature difference between the particle and the mean flow :

$$f = -g\rho\,\frac{(\mathrm{d}\rho - \delta\rho)}{\rho} = -g\rho\,\frac{(\delta T - \mathrm{d}T)}{T} \qquad (30)$$

where $\mathrm{d}\rho$ (or $\mathrm{d}T$) is the variation of the density (or temperature) of the fluid particle as it moves while

$\delta\rho$ (or δT) is the density variation of the fluid surrounding the particle. The thermal dissipation, as it tends to relax this temperature difference, decreases the temperature amplitude of the backward force exerted on the particle and limits the stabilizing effect of the buoyancy force.

That is the case when there is a Newtonian cooling. Let us consider a fluid particle which moves up from z to $z + \delta z$; while its density change is $d\rho$, its temperature change is dT and its pressure change is $p_z \, \delta z$.

When there is a Newtonian cooling b, the entropy change is :

$$\frac{dS}{dt} = -bC_p\left(\frac{dT - \delta T}{T}\right). \qquad (31)$$

If we introduce a time scale t_c representative of the phenomenon which displaces the fluid particle, we can roughly express the thermal exchange law as :

$$\frac{dT}{T} - k\frac{dp}{p} = -bt_c\left(\frac{dT}{T} - \frac{\delta T}{T}\right). \qquad (32)$$

Furthermore, we have :

$$\frac{dp}{p} - \frac{d\rho}{\rho} = \frac{dT}{T}. \qquad (33)$$

We can derive from this relation the buoyancy force :

$$f = g\,(d\rho - \delta\rho) = -g\rho\,\frac{\bar{\theta}_z}{\bar{\theta}} \times \frac{1}{1 + bt_c}. \qquad (34)$$

If we compare it to the value obtained in an adiabatic situation :

$$f_a = -g\rho\,\frac{\bar{\theta}_z}{\bar{\theta}}. \qquad (35)$$

We see that the Newtonian cooling decreases its amplitude and then limits its stabilizing effect.

These changes in the stability characteristics of the flow are not surprising since the usual stability theorem is not valid when there is dissipation (Miller and Lindzen, 1988).

When there are Newtonian cooling or Rayleigh friction, it is possible to derive a stability theorem in a way similar to that used to derive the wellknown Miles Howard's theorem (Miles, 1961 ; Howard, 1961). The necessary condition for the existence of an instability which growth rate is kc_i is that :

$$\frac{N^2}{U_z^2} < \frac{1}{4}\,\frac{\left|c_r - U - i\left(c_i + \frac{b}{k}\right)\right|^2\left(c_i + \frac{a}{k}\right)}{\left|c_r - U - i\left(c_i + \frac{a}{k}\right)\right|^2\left(c_i + \frac{b}{k}\right)} \qquad (36)$$

must be valid somewhere in the flow. b is the Newtonian cooling while a is the Rayleigh friction. It is possible to derive this result in the hydrostatic as well as in the Boussinesq approximation.

For $a = b$, we find again the usual necessary stability condition.

For $b > a$, unstable modes ($c_i > 0$) may exist in a flow having a minimum Richardson number greater than 0.25.

In fact, we can have $R_i > 1/4$ and verify :

$$R_i < \frac{1}{4}\,\frac{c_i + b/k}{c_i + a/k}$$

at the critical level. These results confirm the assumption that the Newtonian cooling can have a destabilizing effect on the flow.

Figure 12 shows that the Rayleigh friction tends to decrease the reflection (and the transmission) of the gravity wave. The form of the solution around the critical level can partly explain this result.

With Rayleigh friction, the Taylor-Goldstein equation for a mode

$$w(z)\, e^{ik(x-ct) + z/2h}$$

in the hydrostatic approximation is :

$$w_{zz} + \left\{\frac{J}{\left(U - c - i\,\frac{a}{k}\right)(U - c)} - \frac{U_{zz} + U_z/h}{\left(U - c - i\,\frac{a}{k}\right)} - \frac{1}{4h^2}\right\} w = 0. \qquad (37)$$

Near the critical level, we have $U - c = z + \cdots$ and we can obtain by a Frobenius expansion the form of the solutions at the critical level :

$$z > 0 \qquad w_1 = z + a_2 z^2 + \cdots \qquad (38)$$

$$w_2 = 1 + 2\,a_2\, z\ell n z + a_2 z + \cdots \qquad (39)$$

$$\text{with} \qquad a_2 = -i\,\frac{J}{a}\frac{k}{2}. \qquad (40)$$

For $z < 0$, we obtain the form of w_2 by an analytical continuation through the critical level :

$$w_2 = 1 + 2\,a_2\, z\ell n|z| + (a_2 - 2\,a_2\, i\pi)\, z + \cdots \qquad (41)$$

We write a general solution $w = Aw_1 + Bw_2$ and we calculate the vertical energy flux :

$$F_w = \text{Real}\;((U - c)\, uw^* + \varphi w^*)/2$$

$$F_w = \text{Real}\left(-\frac{a}{2k^2}\left(w_z - \frac{w}{h}\right) w^*\right) \qquad (42)$$

and we find that there is a loss of energy flux at the critical level :

$$F_w(0^+) - F_w(0^-) = -\frac{\pi}{2}\frac{J}{k}\,|B|^2. \qquad (43)$$

The Newtonian cooling gives a continuous energy flux through the critical level : $F_w(0^+) - F_w(0^-) = 0$.

We obtain the same results in the Boussinesq approximation.

As the reflection and transmission of the wave are closely related to the energy budget of the perturbed field, the energy loss observed can be associated with the decrease in the reflection and transmission coefficient obtained in the numerical experiments when there is Rayleigh friction. We presume that the Rayleigh friction has a stabilizing effect on the flow.

To confirm this assumption, we test the influence of the Rayleigh friction on the stability problem of

Drazin (Fig. 13). This figure shows that the Rayleigh friction tends to strongly move down the neutral stability curve.

On the problems of wave reflection and Drazin instability, the Rayleigh friction tends to have a stabilizing influence which is qualitatively similar to that observed in Sections 3 and 4*b* when viscosity was introduced.

Furthermore, in Sections 3 and 4, we observed that the changes in the thermal conduction does not influence these two problems. This seems to be a very different effect to that observed when Newtonian cooling is introduced. The main reason is that we limit our investigations to Prandtl number greater than 0.1 and lesser than 10. Nevertheless, Jones (1968) and Miller and Lindzen (1988) found, for very low values of the Prandtl number, unstable modes which don't exist in the inviscid case. Numerically, we observe that for $Pr \simeq 10^{-2}$-10^{-3}, unstable modes can appear in the Drazin problem when $J > 1/4$ and that the reflection and transmission of a gravity wave propagating through a critical level can be greater than those obtained in the inviscid case. Nevertheless, this destabilizating influence of the thermal diffusion is very irregular. For instance, at a given Reynold number, we observe that the reflection and transmission coefficients of a gravity wave propagating through a critical layer decrease a little when the thermal diffusion increases and reach important values for very low Prandtl number. Similar irregularities are observed when we calculate the changes in the growth rate of an unstable mode in the Drazin problem. In these two cases, the changes are monotonic when the Newtonian cooling increases and for a constant Rayleigh friction.

This difference is not surprising because the Newtonian cooling and the thermal conduction act very differently in the processes described by Jones : the Newtonian cooling relaxes the temperature differences while the thermal conduction relaxes the temperature gradient differences.

In this paper, we limit our study to $Pr \simeq 0(1)$ because, even when eddy dissipation exists, it is not realistic to consider very low values of the Prandtl number in the atmosphere. In this case, the influence of the viscosity dominates that of the thermal diffusion.

Nevertheless, in the problem of outgoing wave instabilities, the thermal diffusion can have a destabilizing effect even when $Pr \simeq 0(1)$. In fact, we see on Figure 9 that the value of the largest pseudo-Richardson number $N_0^2 d^2/u_0^2$, at which resonant over-reflection arises, increases as the Prandtl number decreases.

6. INFLUENCE OF VISCOSITY AND OF THERMAL CONDUCTION WHEN THEY ARE CONFINED AROUND THE CRITICAL LEVEL

The values of the parameters which can influence reflection, transmission and instabilities are those found in the high mesosphere assuming that the dissipation is caused by molecular viscosity and heat conduction.

Let us assume that eddy viscosity and eddy conduction of heat can be properly described by the same mathematical formalism. The main difference in considering eddy dissipations is that the phenomenon will be confined to a layer surrounding the critical level. It means that the physical properties of the medium change more or less abruptly near this level. Passing from one region to another, the waves must undergo reflection and can be partially trapped inside the viscous layer. The interaction between gravity waves and turbulence has been extensively studied by Fua and Einaudi (1984), Einaudi and Finnigan (1981) and Finnigan and Einaudi (1981). Our interest is to study the influence of a turbulent layer on the reflection and transmission of gravity waves and also on the instability characteristics of the Kelvin-Helmholtz waves.

The numerical results indicate that there is always an optimal thickness of the dissipation layer for which the reflection and the transmission of the wave have at least a relative maximum value.

The wave being trapped, this maximum must be related to the relative phase of the successive reflections inside the layers.

If the values of the viscosity and of the thermal diffusion are increased, the reflectivity of these « boundaries » can be increased, but the dissipative effects of the critical level become more important and compensate the former effect.

In Section 3, we showed that the thermal conduction has little effect on the reflection of the wave at the critical level when $Pr \simeq 1$. We can expect that for values of the Prandtl number less than 1, the dissipative layer will increase the reflection of the wave in regards to the inviscid case. The results reported in Figure 14 agree with this assumption.

For $Pr = 10$, the wave reflection has a maximum smaller than in the inviscid value. For $Pr = 3$, the reflection coefficient is also smaller than in the inviscid case.

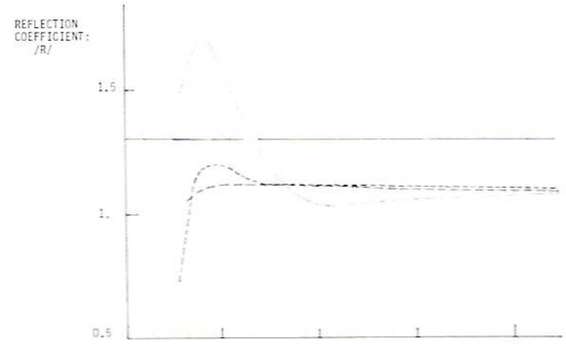

Figure 14

Variation of the reflection coefficient as a function of the depth of the viscous layer surrounding the critical level. $U(a) = 1 + kz$; $n^2(z) = 1$; $c = 1$; $kRe = 83$. $J = 0.1$, *the straight line* (——) *corresponds to the inviscid value of the reflection coefficient.* (……) $Pr = 0.1$; (-·-·-) $Pr = 1$; (------) $Pr = 10$.

The curve corresponding to $Pr = 0.1$ shows what we expect. The reflection is greater than in the inviscid case and it decreases then to lower values indicating that we are in the presence of a partially trapped wave.

Similar simulations concerning the Drazin stability problem show that when the viscosity and the thermal conduction are confined around the critical level, unstable modes appear for values of the minimum Richardson number which are greater than 0.25. As in the preceding problem, this last result strongly depends on the Prandtl number.

7. CONCLUSION

In this paper, we studied the influence of dissipative processes on the reflection and transmission of a gravity wave propagating through a shear layer as well as on some stability problems.

We found that for high Reynolds number when values of the viscosity are consistent with those found in the middle atmosphere, the reflection and transmission of a wave incident upon a shear layer are the same as in the inviscid case. These results confirm previous theoretical results found when the background wind profile is an hyperbolic tangent or a jet containing two critical levels. These results have been obtained with a linear approximation which is assumed to be valid for sufficiently small wave amplitudes. To define more precisely these limitations, we test the incidence of non-linearities with the help of a fully non-stationary non-linear model. We observe that for low wave amplitudes ($|u| < 30\ \mathrm{cm\ s^{-1}}$ near the critical level, for instance), the reflection and transmission coefficients obtained in linear and in non-linear simulations are very similar during a very long time (at least 20 wave periods). If we impose higher amplitudes to the forced wave, secondary modes can grow and the background wind undergoes rapid modification. In this case, the system cannot reach a stationary state. To a certain extent, this non-linear phenomen can be reduced when the values of the viscosity and of the thermal conduction increase. Nevertheless, this last effect is limited, and we observe that the linear approximation is never valid if the amplitude of the incident wave exceeds a few $10\ \mathrm{cm\ s^{-1}}$ while the maximum horizontal wind induced by the wave near the critical level does not exceed $1\ \mathrm{m\ s^{-1}}$ when the velocity scale of the flow is $10\ \mathrm{m\ s^{-1}}$.

The instabilities we study are those which in the limit of vanishing growth rate are evanescent waves and those which in this limit are propagating waves.

Typical instabilities of the first type are those discovered by Drazin (1968) for which we find that the maximum Richardson number allowing neutral solutions decreases linearly with increasing viscosity. This decrease found numerically is more rapid than that found analytically by Churilov and Shukhman (1987). Here the validity of the linear approximation is limited to rather small amplitudes.

Instabilities of the second type are those which can exist if the stratification vanishes at the critical level. This situation can exist when a turbulent layer has been developed at this place. Then, resonant over-reflection can occur in the inviscid case.

Going further in this analysis, we discover that for some background conditions, the fluid becomes unstable for modes which in the limit of vanishing growth rate are propagating waves. The influence of viscosity is very surprising in that it allows the existence of instabilities for pseudo-Richardson number higher than without dissipations. In this case also, the linear approximation is valid for low values of the wave amplitude ($|u| \simeq 0.2\ \mathrm{m\ s^{-1}}$). Furthermore, the non-linear model shows that the wave can reach higher amplitude but with a time growth which is linear with respect to the time instead of being exponential.

We studied also the effect of Rayleigh friction and Newtonian cooling which are often used to parameterize viscosity and thermal conduction.

The stabilizing influence of Rayleigh friction we observe in Drazin's instability problem and in the gravity wave critical level interaction problem looks like that observed when viscosity is introduced. It is meaningful to note that, concerning the problem of reflection and transmission of the wave when there is Rayleigh friction, we can calculate a loss of energy flux at the critical level.

The destabilizing influence of the Newtonian cooling we observe can also be compared to that of thermal conduction. However, when we consider realistic values of the Prandtl number (i.e. $Pr \simeq 1$), the effect of the thermal dissipation is dominated by the stabilizing influence of the viscosity. Furthermore, the changes in the properties of the phenomena we study when the thermal diffusion increases (and Re remains constant) are very irregular compared to those obtained when the Newtonian cooling increases (and the Rayleigh friction remains constant). This difference can be due to the fact that those thermal dissipations act very differently on the perturbed temperature field.

These changes in the stability characteristics of the flow are not surprising since the Miles Howard theorem is not valid when there is dissipation. It is meaningful to note that we can derive again this stability theorem when there are Rayleigh friction and Newtonian cooling. We observe that the necessary condition for the existence of unstable modes is no longer valid and that we can have unstable modes in a flow where $Ri > 1/4$ everywhere if the Newtonian cooling is greater than the Rayleigh friction.

Furthermore, we showed that dissipative processes, when they are confined near the critical level, as it can be if a turbulent layer is present, the reflection and transmission of the wave as well as the Richardson number allowing instabilities in the Drazin problem, can increase. This is a consequence of the limits of the dissipative region acting as partially reflecting boundaries.

Most of the results in the present study concern the linear approximation. We have shown that such an approximation remains valid for low amplitude waves. In a next study, we will investigate the non-linear properties of these problems when the amplitudes are larger and when there is dissipation.

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