Geophys. Astrophys. Fluid Dynamics, Vol. 66, pp. 133-167
Reprints available directly from the publisher
Photocopying permitted by license only

NONLINEAR DISSIPATIVE CRITICAL LEVEL INTERACTION IN A STRATIFIED SHEAR FLOW: INSTABILITIES AND GRAVITY WAVES

FRANÇOIS LOTT and HECTOR TEITELBAUM

Laboratoire de météorologie dynamique du C.N.R.S., Ecole Polytechnique, 91128 Palaiseau Cedex, France

(*Received 4 September 1991; in final form 23 January 1992*)

A two-dimensional numerical model is used to investigate the nonlinear dissipative interaction between a disturbance and an unbounded stratified shear flow. The disturbances considered are Kelvin-Helmholtz instabilities and forced gravity waves. The nonlinear stabilization (destabilization) of Kelvin-Helmholtz instabilities at Prandtl number, $Pr<1$ ($Pr>1$), found by Brown *et al.* (1981) is recovered. The model confirms that it is mostly due to a stabilization (destabilization) of the mean flow by the wave. The nonlinear evolution of instabilities existing when thermal dissipation is large and when the Richardson number is everywhere larger than 0.25 is also investigated. It is shown that such a mode stops growing when the nonlinear distortion of the mean flow becomes significant.

For forced gravity waves, and when the initial minimum Richardson number, $J=0.25$, it is found that mean flow stabilization (destabilization) also occurs at the critical level for $Pr<1$ ($Pr>1$). More generally, the value of Pr above (below) which critical level destabilization (stabilization) occurs increases (decreases) when J increases (decreases). The nonlinear reflection and transmission of a wave are partly related to those stability changes. They are also related to the mean flow distortions, located below the critical level, and where the incident wave can be strongly reflected.

In the present study, the critical level interaction is weakly nonlinear, quasi-steady and dissipative. The amplitude of the fundamental mode is such that the nonlinear effects are significant while secondary modes remain small and convective overturning does not occur.

KEY WORDS: Gravity waves, Kelvin-Helmholtz instabilities, nonlinear critical layer, dissipation, Prandtl number.

INTRODUCTION

The interaction between a disturbance and the mean flow at a critical level is known to have important consequences in the dynamics of the atmosphere and the ocean. For unbounded stratified shear flows, these disturbances are either spontaneously generated at the shear layer or forced outside of it. We shall name the former instabilities, and the later gravity waves even if, in some cases, the differences are not sharp. Linear inviscid theory for such disturbances has shown that their behaviour depends on the Richardson number at the critical level. If the Richardson number is larger than 0.25 everywhere in the flow, unstable modes cannot exist (Miles, 1961; Howard, 1961); they can arise for minimum Richardson number smaller than 0.25 (Drazin, 1958). Gravity waves are absorbed at the critical level if the Richardson number there is larger than 0.25 (Booker and Bretherton, 1967), otherwise they can be over-reflected (Jones, 1968). The fact that the behaviour of both instabilities and gravity waves is indicated by the Richardson number suggests that these perturbations are somewhat connected. Lindzen and Rosenthal (1983) have shown that Kelvin-Helmholtz instabilities result in physical processes which are similar to those involved in the wave over-reflection. A Kelvin-Helmholtz instability can be interpreted as the superposition of over-reflected vorticity waves trapped at the shear layer (Lindzen and Rosenthal, 1983). On the other hand, an over-reflected gravity wave can become an unstable mode if it is reflected back towards the shear layer by a rigid wall (or by a turning height beyond which it is evanescent) (Rosenthal and Lindzen, 1983). Such a mechanistic picture fails for the Kelvin-Helmholtz instability occurring when the Richardson number at the critical level is larger than 0.25 and goes to zero outside the shear layer (Smyth and Peltier, 1989). It also fails for gravity waves spontaneously radiating away from the critical level, which are unstable modes occurring when the stratification vanishes at the critical level (Lott and Teitelbaum, 1990). Both kinds of instabilities are out of the scope of the present study since we consider a constant stratification in the whole unbounded domain.

Nevertheless, it must be pointed out that the linear inviscid approximation is only valid in the limit of very small amplitude disturbances. More precisely, the regime of the critical level depends on the relative size of three scales: the viscous scale, l_v, the unsteady scale, l_t, and the nonlinear scale, l_n, (Churilov and Shukhman, 1987) defined as:

$$l_v=(k\,Re)^{-1/3}, \qquad l_t=\partial_t\varepsilon/k\varepsilon, \qquad l_n=\varepsilon^p.$$

Here p varies from $\frac{1}{2}$ to $\frac{2}{3}$, when J varies from 0 to 0.25 and $p=\frac{2}{3}$ for $J>0.25$ (see Appendix A for details). We also introduce a thermal diffusion scale l_{th}:

$$l_{th}=(k\,Pr\,Re)^{-1/3}.$$

In this study, all the quantities used are normalized by the characteristic parameters of the shear layer (the maximum velocity amplitude, U_0, the shear layer depth, d, and the constant Brunt Väisälä frequency, N_0), k is the horizontal wave number of the disturbance and ε characterizes its amplitude; Re is the Reynolds number and Pr is the Prandtl number measuring the ratio between the viscosity and the thermal conduction. The lengths l_v (l_{th}), l_t and l_n characterize the normalized depths of the layers, surrounding the critical level, where the viscosity (the thermal conduction), the unsteadiness or the non-linearities are important. For Kelvin-Helmholtz instabilities, the unsteady scale is related to the linear growth rate of the unstable mode:

$$\gamma=\partial_t\varepsilon/\varepsilon.$$

For a gravity wave, the unsteady scale is associated to the temporal variations of the amplitude, S, of the wave source:

$$l_t=\partial_t|S|/k|S|.$$

For continuously stratified shear flow, the linear unsteady critical level has been examined by Howard (1963), who calculated the growth rates of unstable modes in the vicinity of neutral stability curves. Viscosity and thermal conduction have been introduced by Maslowe and Thomson (1971) for linear Kelvin-Helmholtz instabilities. They have shown that for large values of the Reynolds number the dominant parameter concerning the stability of the mean flow remains the Richardson number. Similar results have been obtained by Hazel (1967) and Van Duin and Kelder (1986) for the reflection and transmission of a gravity wave incident upon a shear layer in presence of dissipation. With large dissipation, the linear approach gives results which significantly depend on the value of Re and Pr. For $Pr \geqslant 1$ the growth rates of instabilities and the reflection and transmission coefficients of a gravity wave are lower than in the inviscid case. Opposite effects can occur at low Prandtl number (Miller and Gage, 1972; Miller and Lindzen, 1988) or very low Prandtl number (Jones, 1977; Lott and Teitelbaum, 1990). This destabilization has been partly interpreted by Jones (1977): "if the stratification is thermal in origin, radiative diffusion will weaken the stabilizing effect of the stratification; and may allow instability to occur". In this case, unstable modes can occur even if the Richardson number is everywhere larger than 0.25.

The nonlinear evolution of an inviscid disturbance at a critical level has been extensively studied for unstable modes as well as for gravity-wave modes [see Stewartson (1981) and Maslowe (1986) for a detailed review of critical level processes]. When the flow is marginally unstable, Brown and Stewartson (1978) have shown that a neutral mode can generate a mode which grows algebraically with time. When the flow is unstable, Churilov and Shukhman (1988) have shown that the amplitude of the most unstable mode nonlinearly grows as $(t_0 - t)^{-7/4}$. Klassen and Peltier (1985a, b), using a numerical model, considered the full development of Kelvin-Helmholtz billows in a strongly unstable flow with weak dissipation. They found that the overturning of the wave occurs earlier when the Prandtl number is larger (Klassen and Peltier, 1985b). Collins and Maslowe (1988), reported a strong instability mechanism involving harmonic resonance of two unstable modes. Nonlinear gravity waves in an inviscid flow were studied analytically by Brown and Stewartson (1980, 1982a, 1982b). They showed that reflection (transmission) of nonlinear gravity waves incident upon a shear layer is larger (smaller) than that of the linear case.

Maslowe (1977) studied the nonlinear dissipative evolution of the Holmböe mixed layer profile at $Pr = 0.72$ and found that subcritical instabilities can occur. More recently, Brown *et al.* (1981) have shown that the critical layer is supercritically stable (unstable) when $Pr < 1$ ($Pr > 1$). They have shown that it is mostly due to stabilization (destabilization) of the mean flow by the wave. These results were confirmed by Churilov and Shukhman (1987) who extended the study to include terms overlooked by Brown *et al.* (1981). The nonlinear dissipative regime of gravity waves has been studied numerically by Clark and Peltier (1984). They showed that large amplitude waves are strongly reflected at the critical level even when the initial flow is stable. Fritts (1978, 1982) reported on the formation of a thin turbulent shear layer associated with unstable regions which appear during gravity wave-critical level interaction. He also discussed the relative importance of viscosity and unsteadiness on the local destabilization of the total flow near the critical level.

In the present study, we consider the problem of weakly nonlinear development of instabilities, as well as the problem of weakly nonlinear reflection and transmission of gravity waves propagating through a critical-level when dissipation dominates unsteadiness [i.e. $\mathrm{Min}(l_\nu, l_{th}) \gg l_t$]. We restrict our attention to relatively small disturbance amplitudes, so that convective overturning doesn't occur. Nevertheless, the amplitudes remain large enough to make the nonlinear scale, l_n, of the same order of magnitude as the smallest dissipative scale (l_ν or l_{th}). Furthermore, the range of parameters used is such that the nonlinear evolution proceeds faster than the viscous spreading of the initial flow; viscous spreading is therefore neglected. However, the diffusion of nonlinear mean flow distortion is taken into account because its small spatial scale makes it sensitive to dissipation (Brown *et al.*, 1981, Churilov and Shukhman, 1987).

The aim of the present study is to describe the mechanisms governing the nonlinear exchanges between a disturbance and the mean flow and to determine the feedback effects of the mean flow changes on the behaviour of the disturbance itself. Such a "quasi linear" description of the critical layer is found to be valid in the range of parameters we consider. For Kelvin-Helmholtz instabilities, the first motivation is to recover the result of Brown *et al.* (1981), using a fully nonlinear finite difference two dimensional time-dependant model. The spatial resolution of our model allows us to examine the details of the nonlinear evolution of the system. The nonlinear stabilization of unstable modes observed at $Pr < 1$ leads us naturally to reconsider the evolution of unstable modes which appear in the linear approximation when $J > 0.25$ and when there is large thermal dissipation. The analysis of the development of Kelvin-Helmholtz instabilities also leads us to investigate the problem of the gravity wave critical layer interaction since some similarities exist in the physical processes governing both phenomena. Furthermore, gravity waves and Kelvin-Helmholtz instabilities can coexist simultaneously at a shear layer. Therefore, the induced nonlinear reflection and transmission of a gravity wave is studied using the same fully nonlinear model. In the present study, the Prandtl number values used do not necessarily correspond to values existing in real fluids. Nevertheless, it must be emphasized that the Prandtl number for geophysical and astrophysical fluids varies widely. The solar atmosphere is a good example of (very) small Pr, because, the dynamic viscosity is determined by the small molecular mean free path whereas the thermal conduction is determined by the large radiation (photon) mean free path. The ocean is an example of large Pr, because $Pr = 6.73$ in water, and the influence of salinity can lead to larger values. In laboratory experiments, when the density variation is provided by variations of salinity, the relevant Pr can be very large ($Pr \approx 500$). The atmospheric value of Pr is 0.73. Unfortunately, since the vertical numerical gridspacing has to be smaller than l_ν and l_{th}, sufficient resolution can only be obtained by restricting maximum and minimum values of Pr. Thus, for the purpose of this study, we restrict consideration to disturbances with Prandtl number varying from 0.1 to 10. Nevertheless, note that the results obtained here when $Pr = 1$, apply within a very good approximation to the atmosphere while those obtained for $Pr \approx 10$, qualitatively apply to the ocean.

The structure of the paper is as follows. In Section 1, the numerical model used in our simulations is presented. Section 2 is devoted to the nonlinear dissipative evolution of a Kelvin-Helmholtz instability. In Section 2.a, we interpret theoretically the nonlinear interaction using simplified dissipation (i.e. Newtownian cooling and Rayleigh friction). These results are qualitatively extended to the case of realistic dissipation and $Pr \neq 1$ in Section 3.b. Numerical results are presented in Section 2.c. In Section 2.d, we extend the analysis to modes existing when thermal dissipation is large and when the Richardson number is larger than 0.25. Section 3 concerns the nonlinear dissipative reflection and transmission of a gravity wave propagating toward a critical level for various minimum Richardson numbers. Following the preceding analysis, Section 3.a deals with the action of the wave on the mean flow using simplified dissipation. The results obtained are qualitatively extended to realistic dissipation and $Pr \neq 1$ in Section 3.b. Numerical results are presented in Section 3.c.

1. THE MODEL

We consider the nonlinear evolution of a disturbance in a stratified shear layer with the vertical profiles (hereafter named as the Drazin profile):

$$\tilde{U}(\tilde{z}) = U_0 \tanh(\tilde{z}/d), \qquad N^2(\tilde{z}) = N_0^2 = constant.$$

The horizontal phase velocity of the disturbance is 0 so that a critical level exists at the inflection point of the mean flow ($\tilde{z}=0$). Introducing d, U_0, d/U_0 and $N_0^2 d$ as units of length, speed, time and buoyancy force, the system of equations is written in dimensionless form. Using the Boussinesq approximation, we introduce a stream function ψ associated with the disturbance and defined as

$$\frac{\partial\psi}{\partial z} = u, \qquad -\frac{\partial\psi}{\partial x} = w.$$

The buoyancy force is

$$\varphi = g(\theta/\theta_0),$$

where θ is the modification of the potential temperature induced by the disturbance and θ_0 is the initial potential temperature. Then, the system of equations of motion is written in the stream function vorticity form

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\Delta\psi - \frac{d^2U}{dz^2}\frac{\partial\psi}{\partial x} + J\frac{\partial\varphi}{\partial x} + a\Delta\psi + \frac{da}{dz}\frac{\partial\psi}{\partial z} - \frac{\Delta\Delta\psi}{Re} = -\{\psi, \Delta\psi\}, \qquad (1.1)$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\varphi - \frac{\partial\psi}{\partial x} + b\varphi - \frac{\Delta\varphi}{Pr\,Re} = -\{\psi, \varphi\} + S, \qquad (1.2)$$

where

$$\{g, h\} = \frac{\partial g}{\partial z}\frac{\partial h}{\partial x} - \frac{\partial g}{\partial x}\frac{\partial h}{\partial z},$$

S is a heating source,
$U(z) = \tanh(z)$ is the normalized initial horizontal mean wind,
$J = N_0^2 d^2/U_0^2$ is the minimum Richardson number of the flow located at $z=0$,
$Re = dU_0/\nu$ is the Reynolds number and ν is the coefficient of dynamic viscosity,
$Pr = \nu/\kappa$ is the Prandtl number, and κ is the coefficient of thermal conduction,
a and b are the normalized Rayleigh friction and Newtownian cooling coefficients respectively. In the numerical model, a and b are used to introduce sponge layers at the top and at the bottom of the field. These sponge layers prevent reflections there and allow the introduction of simple boundary conditions

$$\psi = 0 \qquad \text{at} \quad z = z_b, z_t.$$

In the horizontal direction, we apply periodic boundary conditions. The heating source, S, is introduced to force an incident gravity wave below the shear layer. It is expressed as a function of z and t as

$$S(t,z) = f(t)\exp\left[-\frac{(z-z_s)^2}{\Delta z_s^2}\right]\exp(ikx), \tag{1.3}$$

where $z_s < 0, |z_s| \gg 1, \Delta z_s \approx O(1)$. Here $f(t)$ is a slowly varying function growing from zero at $t=0$ and reaching a constant value after a time $t_0 \approx O[(kl_t)^{-1}]$ which is long enough to limit the amplitude of transient effects. When unstable modes are studied, we set $S=0$ and some noise is introduced in the field at the initial time. During the temporal evolution, it is natural to separate the disturbance between the mean flow deviation and the wave as

$$\varphi = \bar{\varphi}(z,t) + \varphi'(z,x,t),$$

where

$$\varphi' = \sum_{-\infty, l\neq 0}^{+\infty} \varphi_l(z,t)\exp(ilkx).$$

When gravity waves are studied, k is the horizontal wave number imposed by the wave source. When instabilities are studied, k is the horizontal wave number of the most unsable mode appearing in the flow. The spatial derivatives in x and z directions are calculated with centred finite differences. In the horizontal direction, a 32-point grid is used for the instabilities studies. We have verified that this resolution is adequate for our purposes. For gravity waves, a higher resolution is necessary, since

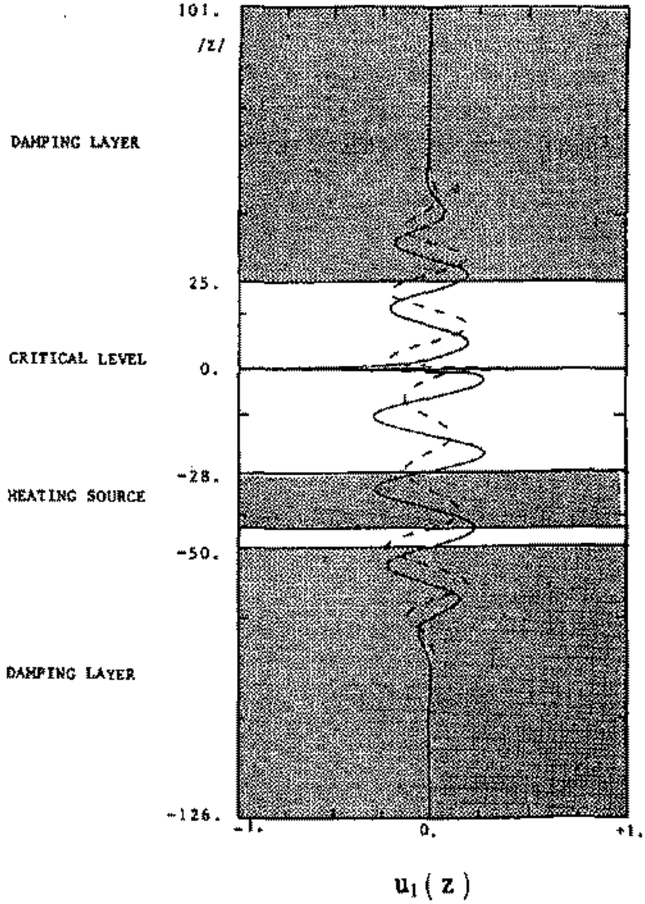

Figure 1 Vertical profile of the horizontal velocity u_1, induced by the fundamental mode of a gravity wave at a given x_0(——) and at $x_0+\pi/2k$ (– – –).

small scale unstable modes as well as long scale forced gravity waves can be present simultaneously. For these simulations, we verified that a 128-point grid is sufficiently accurate. Since particularly fine vertical resolution is required near the critical level, we use a stretched grid in the vertical direction. At the critical layer, the step length is 0.02 normalized units, it is smaller than the smallest viscous scale (l_v or l_{th}). Our numerical experiments reveal that this vertical resolution was sufficient for convergence of the solution. However, the use of a stretched grid introduces numerical complications and requires caution. Specifically, far from the critical layer, as the grid spacing becomes large, we are not able to resolve modes due to viscosity and thermal conduction. Furthermore, changes in grid spacing can induce numerical reflections and waves can be trapped in the high resolution zone (i.e. the critical layer). Nevertheless, comparison of results obtained when a stretched grid was used, with those obtained when the grid spacing was small and constant in the whole field, shows similar time evolution of the flow.

The temporal integrations of $\Delta\psi$ and φ are performed with a predictor corrector algorithm used by Lindzen and Barker (1985). The vertical integration of the stream function is calculated by a Gaussian elimination technique. Figure 1 illustrates the propagation of a gravity wave in the field described above. The wave induced horizontal velocity $u_1(x, z, t)$ is displayed on the figure at a given time t_0 and at two different locations x_0 and $x_0 + \pi/2k$. In order to characterize the importance of the nonlinear terms in the propagation of the disturbance, we employ two versions of the model: the fully nonlinear version (equations 1.1 and 1.2) and a linear version wherein the terms $\{\psi, \Delta\psi\}$ and $\{\psi, \varphi\}$ are neglected. Results of the linear calculations are also used to quantify the action of the wave on the mean flow before significant nonlinear distortion occurs.

2. INFLUENCE OF DISSIPATION ON INSTABILITY NONLINEAR EVOLUTION

In their study of the nonlinear evolution of a Kelvin-Helmholtz instability, Brown *et al.* (1981) calculated the first nonlinear coefficient, α, of the Landau equation when the critical layer is controlled by dissipation

$$\frac{dA}{dt} = \gamma A + \alpha A|A|^2.$$

Here γ is the linear growth rate of the most unstable mode existing in the flow and A is its amplitude (it is related to the parameter ε through $\varepsilon = |A|$). They showed that the value of α strongly depends upon the Prandtl number. Specifically, when $Pr > 1$, α is positive and the growth of the instability increases nonlinearly. When $Pr < 1$, the opposite is observed. Brown *et al.* (1981) also reported that α is mostly sensitive to the nonlinear modification of the mean flow stability. They found that the influence of the secondary modes is weak. These results were confirmed by Churilov and Shukhman (1987) who corrected the treatment of mean flow distortion by Brown *et al.* (1981).

2.a Rayleigh friction and Newtownian cooling ($Pr = 1$)

To first gain physical insight into the modification of the mean flow stability induced by the wave, we consider a Kelvin-Helmholtz mode in a weakly unstable flow where

$$\mu = 0.25 - J \ll 1,$$

and dissipation represented by Rayleigh friction and Newtownian cooling. When the

Prandtl number, $Pr=b/a=1$, the equations which drive the mean flow are

$$\frac{\partial \bar{u}}{\partial t}+a\bar{u}=-\frac{d(\overline{u'w'})}{dz}\approx-\frac{1}{2}\frac{d}{dz}[\mathrm{Re}(w_1^* u_1)], \tag{2.1}$$

$$\frac{\partial \bar{\varphi}}{\partial t}+a\bar{\varphi}=-\frac{d(\overline{\varphi'w'})}{dz}\approx-\frac{1}{2}\frac{d}{dz}[\mathrm{Re}(w_1^* \varphi_1)]. \tag{2.2}$$

The forcing terms in these equations are calculated from the structure of the most unstable mode existing in the flow. Assuming that the horizontal wave number of the disturbance is k, when the nonlinear changes are small, this mode satisfies the linear equations

$$(\gamma+a+ikU)\Delta\psi_1+2ikU(1-U^2)\psi_1+\left(\frac{1}{4}-\mu\right)ik\varphi_1=0, \tag{2.3}$$

$$(\gamma+a+ikU)\varphi_1-ik\psi_1=0, \tag{2.4}$$

and the boundary condition $\psi_1(z\to\pm\infty)\to 0$.
Using these equations, it is straightforward to deduce the growth rate, γ, of the damped mode from the inviscid growth rate, γ_{inv}, by $\gamma=\gamma_{inv}-a$. Howard (1963) showed that the growth rate of the most unstable mode in the Drazin profile is $\gamma_{inv}=k\mu$, and its horizontal wavenumber satisfies $k^2=0.5$. Under these conditions, the growth rate of the damped mode is $\gamma=k\mu-a$.

Far from the critical level, $z\gg\mu$ (the outer region), the structure of the solution can be approximated to first order in μ expanding the solution as a series:

$$\psi_1=\psi_1^1+\mu\psi_1^2+\cdots, \qquad \varphi_1=\varphi_1^1+\mu\varphi_1^2+\cdots,$$

and neglecting the terms of order μ in (2.3) and (2.4). Then, ψ_1^1 is given by the Taylor Goldstein equation

$$L(\psi_1^1)=\Delta\psi_1^1+\left[\frac{1}{4U^2}+2(1-U^2)\right]\psi_1^1=0,$$

with boundary conditions: $\psi_1^1\to 0$ for $|z|\to+\infty$.
The solution (Drazin, 1958) is

$$\psi_1^1=A^{\pm}(t)\frac{\sinh^{1/2}|z|}{\cosh(z)}, \tag{2.5}$$

and $A^{\pm}(t)=A_0^{\pm}\exp[(k\mu-a)t]$.

The ratio A^+/A^- is determined by connecting the upper and the lower half plane through the critical level.

For $z \approx O(\mu)$, the preceding development is invalid since the advection terms in (2.3) and (2.4) are balanced by the damping terms. It is necessary to solve these equations in the inner layer. In the inner layer, the unstable mode satisfies at the first order in μ the equations

$$(y-i)^2 \frac{\partial^2 \psi_1^I}{\partial y^2} + \frac{\psi_1^I}{4} = 0, \qquad \varphi_1^I = \frac{\psi_1^I}{\mu(y-i)}, \tag{2.6}$$

where $z = \mu y$. A solution of (2.6) is

$$\psi_1^I(y) = A(y-i)^{1/2}.$$

Expanding this solution for $y \to \pm\infty$ and expanding the outer solution (2.5) for $z \to 0$ both recover if

$$A = \mu^{0.5} A^+ = i\mu^{0.5} A^-.$$

Thus, an uniformly valid approximation of the most unstable mode is obtained.

The forcing terms in (2.1) and (2.2), at the order we consider, are zero away from the inner layer, since the vertical structure of the outer solution (2.5) corresponds to that of a steady undamped mode. In the inner layer, the action of the wave on the mean flow is

$$-\frac{1}{2}\mathrm{Re}\frac{d(w_1^* u_1)}{dz} \approx -\frac{1}{2\mu^2}\mathrm{Re}\left[ik\frac{d}{dy}\left(\psi_1^{I*}\frac{\partial \psi_1^I}{\partial y}\right)\right] \approx -\frac{k}{4\mu^2}\frac{|A(t)|^2}{(y^2+1)^{3/2}}y,$$

$$-\frac{1}{2}\mathrm{Re}\frac{d(w_1^* \varphi_1)}{dz} \approx -\frac{1}{2\mu^2}\mathrm{Re}\left[ik\frac{d}{dy}\left(\psi_1^{I*}\frac{\partial \psi_1^I}{\partial y}\right)\right] \approx -\frac{k}{2\mu^2}\frac{|A(t)|^2}{(y^2+1)^{3/2}}y.$$

These forcing terms are displayed as a function of z in Figure 2 for $\varepsilon = |A| = 1$ and $\mu = 0.02$. Integrating the mean flow equations (2.1) and (2.2) leads to

$$\bar{\varphi} = 2\bar{u} = -\frac{k}{2\mu^2}\frac{|A(t)|^2}{(y^2+1)^{3/2}}\frac{y}{2k\mu - a}.$$

The wave takes horizontal momentum and potential temperature from the mean flow above the critical level and restores them below. Consequently, the critical level doesn't move and both the mean stratification and velocity shear decrease at the critical level. The associated nonlinear mean flow stability changes around the critical level are given by

$$(\delta Ri)_{z\approx 0} \approx J\left(\frac{d\bar{\varphi}}{dz} - 2\frac{d\bar{u}}{dz}\right)_{z=0} \approx 0.$$

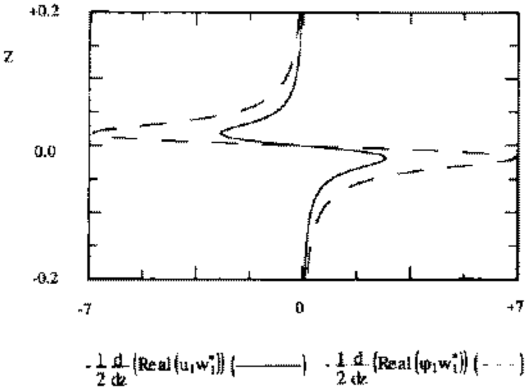

Figure 2 Action of the most unstable Kelvin-Helmholtz mode on the mean flow: $J=0.23$, $k^2=0.5$ and $\varepsilon=1$.

On one hand, the wave has a destabilizing influence on the mean flow because it decreases the stratification. On the other hand it has a stabilizing influence because it decreases the mean velocity shear. When $Pr=1$ (or when an unstable mode grows without dissipation), both effects compensate one another: the Richardson number at the critical level is unchanged through weakly nonlinear dynamics. Furthermore, in the inner layer, the transfers of energy due to these processes are (see Appendix B for a complete presentation of the energetic balance):

$$C(\bar{P},K')=\left\langle -Jz\frac{d(\overline{\varphi'w'})}{dz}\right\rangle \approx \left\langle -\frac{k}{8\mu}\frac{A(t)^2}{(y^2+1)^{3/2}}y^2\right\rangle <0,$$

$$C(\bar{K},K')=\left\langle \overline{u_T}\frac{d(\overline{u'w'})}{dz}\right\rangle \approx \left\langle \frac{k}{4\mu}\frac{|A(t)|^2}{(y^2+1)^{3/2}}yU(y)\right\rangle >0, \quad \text{since } U(y)\gtrless 0 \quad \text{for } y\gtrless 0.$$

In order to grow and to compensate for the wave energy loss due to damping, the wave takes kinetic energy from the mean flow through the downward transport of horizontal momentum. This results in the mean wind shear decrease aforementioned. Part of this kinetic energy returns to the mean flow as potential energy through the downward transport of potential temperature by the wave. This results in the mean stratification decrease aforementioned.

These features are also found when $Pr\neq 1$. However, in this case, the depths of the dissipative layers in which thermal and dynamic exchanges can occur are different. The decrease of the mean flow stratification no longer balances the decrease of the mean velocity shear at the critical level and Ri must change. The same behaviour is also obtained from numerical calculation when the dissipative parameters are viscosity and thermal conduction.

2.b Viscosity and thermal conduction ($Pr \neq 1$)

In presence of viscosity and thermal conduction, the equations governing the mean flow changes induced by the wave are

$$\frac{\partial \bar{u}}{\partial t} - \frac{1}{Re}\frac{d^2\bar{u}}{dz^2} = -\frac{d(\overline{u'w'})}{dz} \approx -\frac{1}{2}\frac{d}{dz}[\mathrm{Re}(u_1 w_1^*)],$$

$$\frac{\partial \bar{\varphi}}{\partial t} - \frac{1}{Pr\,Re}\frac{d^2\bar{\varphi}}{dz^2} = -\frac{d(\overline{\varphi'w'})}{dz} \approx -\frac{1}{2}\frac{d}{dz}[\mathrm{Re}(\varphi_1 w_1^*)].$$

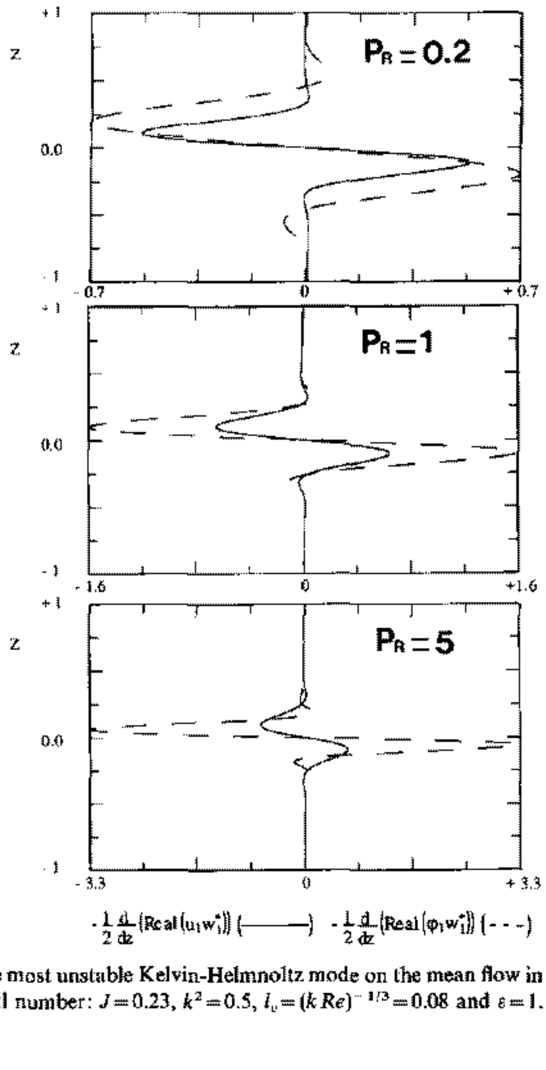

Figure 3 Action of the most unstable Kelvin-Helmnoltz mode on the mean flow in presence of dissipation and for various Prandtl number: $J=0.23$, $k^2=0.5$, $l_v=(k\,Re)^{-1/3}=0.08$ and $\varepsilon=1$.

The forcing terms of these equations are calculated numerically using the linear version of the model presented in Section 1. They are shown in Figure 3 when the amplitude of the wave far away the shear layer is 1 ($\varepsilon=|A|=1$). The parameters are

$$J=0.23;\ k^2=0.5;\ l_v=0.08,\ Pr=0.2,\ 1 \text{ and } 5;\ \gamma_{\rm lin}=0.013\ (l_t=\gamma_{\rm lin}/k=0.018).$$

The linear growth rate, $\gamma_{\rm lin}$, of the unstable mode is the same for the three values of the Prandtl number, since the dissipation is small. The vertical flux convergences are close to those obtained analytically when Rayleigh friction and Newtownian cooling are used (see Figure 2). Figure 3 shows that the depth of the layers in which thermal and dynamic forcing occur are approximately $5l_{th}$ and $5l_v$, respectively. In these numerical experiments, thermal diffusion varies while viscosity remains constant. Then, when the thermal diffusion decreases, the effect of the wave on the mean flow stratification increases because it depends on the vertical second derivative of the thermal flux. Numerical results show that

$$\text{For}\quad Pr\begin{cases}<1\\>1\end{cases},\quad -2\left(\frac{d^2}{dz^2}\left[\mathrm{Re}\left(\frac{u_1 w_1^*}{2}\right)\right]\right)_{z\approx 0}\begin{Bmatrix}<\\>\end{Bmatrix}-\left(\frac{d^2}{dz^2}\left[\mathrm{Re}\left(\frac{\varphi_1 w_1^*}{2}\right)\right]\right)_{z\approx 0}$$

This is also evident by inspection of Figure 3. In this preliminary analysis, we do not consider the diffusion of the mean flow changes. Nevertheless, it can be easily understood that it acts to reinforce the above results. In fact, an increase in a dissipative parameter increases the (negative) slope of the mean flow changes by increasing the depth of the forcing layer and by increasing its diffusion. Consequently, as we shall show:

$$\text{For}\quad Pr\begin{cases}<1\\>1\end{cases},\quad (\delta Ri)_{z\approx 0}\approx J\left(\frac{d\bar{\varphi}}{dz}-2\frac{d\bar{u}}{dz}\right)_{z\approx 0}\begin{cases}>0 & \text{mean flow stabilization}\\<0 & \text{mean flow destabilization}\end{cases}$$

2.c Nonlinear evolution of the instability

Figure 4 displays the temporal variation of the amplitude, ε, of a Kelvin-Helmholtz instability for three different values of the Prandtl number as compared to the one obtained from a linear simulation. The time is represented in units of Doppler shifted periods (i.e. one period: $T=2\pi/k$) of the wave. The amplitude, ε, is determined through a spectral decomposition of the stream function ψ' of the solution at $z\gg \mathrm{Max}(l_v, l_{th})$, where it is verified that the vertical structure of the fundamental mode is approximately given by (2.5). In the nonlinear cases and for very small amplitudes (i.e. $\varepsilon<0.01$ corresponding to $l_n<0.04$), the nonlinear evolutions are similar to the linear one. Thereafter, the amplitudes in the simulations for which $Pr\neq 1$ start to differ from the linear one. As could be predicted by our previous analysis and in agreement with the results of Brown *et al.* (1981) the amplitude ε becomes larger than the linear one for $Pr=5$ and smaller for $Pr=0.2$. At $\varepsilon=0.02$ (i.e. $l_n\approx l_v=0.08$), the three nonlinear temporal evolutions differ significantly. Figure 5 represents the

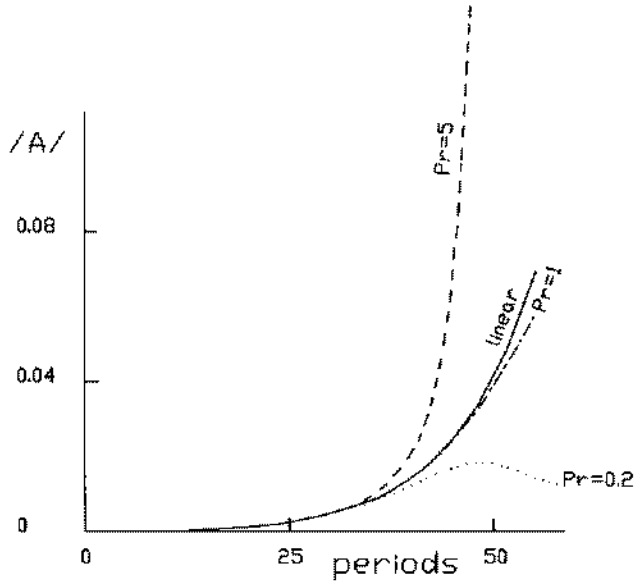

Figure 4 Evolution of the amplitude of the most unstable Kelvin-Helmholtz mode for various Prandtl numbers. The parameters are those of the Figure 3.

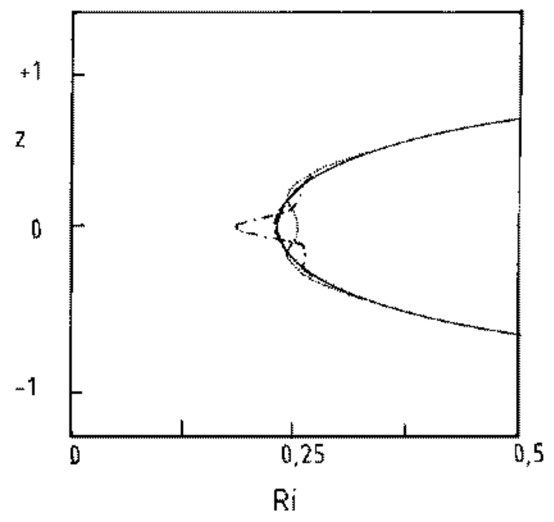

Figure 5 Nonlinear deformation of the stability of the mean flow as a function of z for $/A/=0.01$. Same parameters as Figure 3: linear (———); nonlinear: $Pr=0.2$ (· · ·); $Pr=1$ (— — —); $Pr=5$ (—·—·).

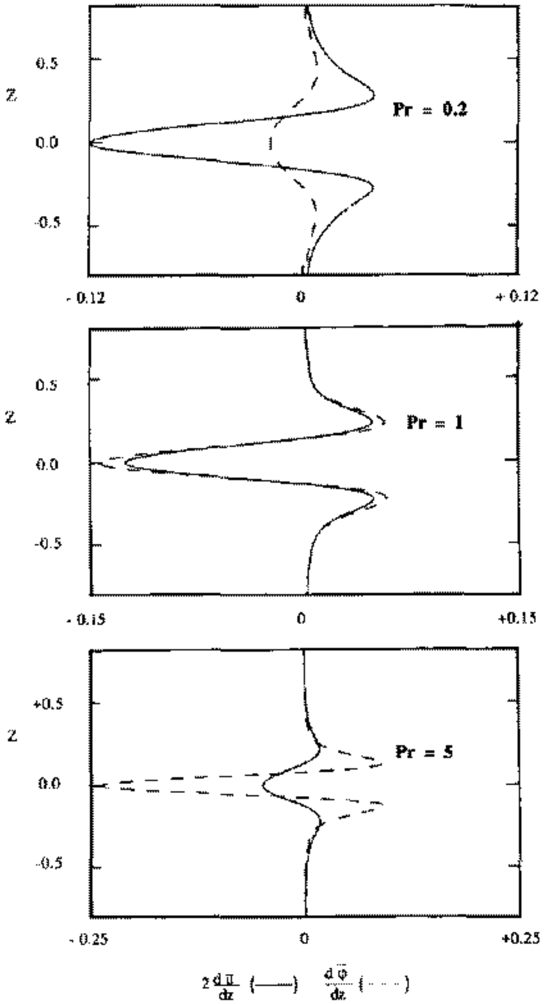

Figure 6 Nonlinear deformation of the mean shears as a function of z for $|A|=0.01$. The parameters are those of Figure 3.

modified vertical profile of the Richardson number when $\varepsilon=0.01$ as compared to the initial one. It is clear that changes in the development of the fundamental mode are due to the mean flow stability modifications. The associated mean shear changes illustrated in Figure 6 show that:

$$\text{For}\qquad Pr\begin{cases}=0.2\\=1\\=5\end{cases},\qquad \left(\frac{d\bar{\varphi}}{dz}\right)_{z\approx 0}\begin{Bmatrix}>\\ \approx\\ <\end{Bmatrix}2\left(\frac{d\bar{u}}{dz}\right)_{z\approx 0}.$$

It recovers the results presented on Figure 4 in Churilov and Shukhman (1987).

The above interpretation of nonlinear evolution concerns the interaction between an unstable mode and the mean flow. Since it does not evoke the presence of secondary modes, it is a quasi-linear interpretation of the phenomena. Brown *et al.* (1981) found that secondary modes change the Landau constant α by less than 20% in a weakly nonlinear regime. In our calculations, as long as the nonlinear scale and the viscous scale have the same order of magnitude, the effects of secondary modes on the nonlinear evolution are negligible. The secondary modes become important just before the occurrence of overturning (i.e. approximately when $l_n > 2l_v$). At $Pr = 0.2$, since the mean flow is nonlinearly stabilized the wave amplitude remains small, and convective overturning does not occur. This last result is characteristic of the dissipative regime: such a nonlinear limitation of the wave amplitude does not exist in the nearly inviscid case (Churilov and Shukhman, 1988). The preceding nonlinear results can also be conveniently represented by the parameter λ proposed by Maslowe (1977) which quantifies the relative importance of nonlinear effects and viscous effects:

$$\lambda = (k\,Re\,\varepsilon^2)^{-1} = (l_v/l_n)^3.$$

In our simulations, we observed that when λ is of order 1 or less than 1, nonlinear concepts are relevant.

2.d *Highly damped modes*

The above results deal with the influence of dissipation on nonlinear instabilities at large Reynolds number. In this case, the condition $l_r \ll l_v$ is very restrictive (at least in the troposphere and the stratosphere) and concerns only weakly unstable flow. Nevertheless when considering molecular dissipation such as it exists in the upper mesosphere, the critical level interaction can remain dissipative for minimum Richardson number significantly lower than 0.25. In this case, we find that the initial nonlinear steps of the temporal evolution of the instability are also stabilized for $Pr < 1$ and destabilized for $Pr > 1$. We verified this for

$$J = 0.2, \qquad Re = 125, \qquad \text{for} \quad Pr = 0.2, 1 \text{ and } 5.$$

Another phenomenon for which the critical level is dissipative is the development of unstable modes in flows with $J \geqslant 0.25$. In the Drazin profile, such modes only occur when the thermal dissipation is very large as compared to the dynamic dissipation. We discuss the nonlinear evolution of such a mode when we use Newtownian cooling as thermal dissipation and zero Rayleigh friction. The parameters of the experiment are

$$J = 0.25; \quad a = 0; \quad b = k/3; \quad k^2 = 0.5; \quad \gamma_{\mathrm{LIN}} = 9.4 \times 10^{-3}.$$

Figure 7 displays the action of this mode on the mean flow. As was true for Kelvin-Helmholtz instabilities, the mode carries mean horizontal momentum downward through the critical level and takes kinetic energy from the mean flow. It tends to decrease the shear $d\bar{u}/dz$ there and to stabilize the mean flow. Part of the kinetic

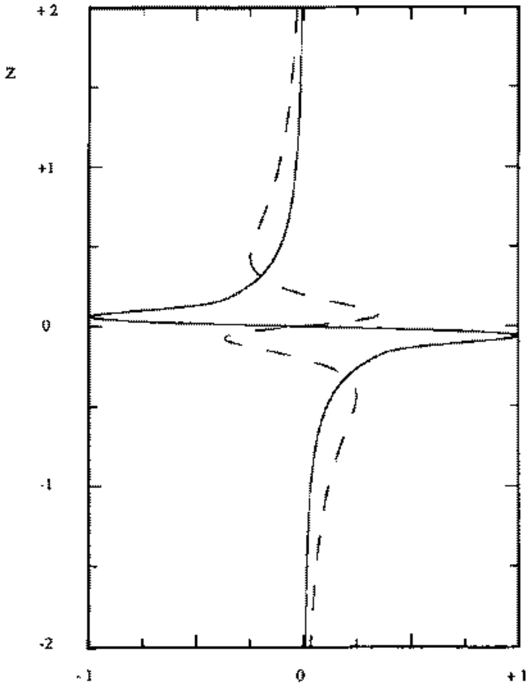

Figure 7 Action on the mean flow of an unstable mode in presence of large Newtownian cooling: $J=0.25$, $k^2=0.5$, $Re=25000$, $Pr=1$, $a=0$, $b=k/3$.

energy taken by the wave is restored to the mean flow as potential energy through a global downward transfer of potential temperature. This behaviour is close to that already discussed. Nevertheless, an important difference occurs in a thin layer at the critical level. The thermal flux locally reverses (it takes potential temperature from the mean flow just below the critical level and deposits it just above) and tends to increase the mean stratification, $d\bar{\varphi}/dz$, there. This stabilizing effect adds to the preceding one even when very large Newtownian cooling tends to smear it out. Figure 8 displays the nonlinear evolution of the amplitude of this mode as a function of time. At the beginning, the amplitude grows exponentially in time as predicted by the linear theory (periods 0 to 35). When the first nonlinear exchanges become significant (periods 35 to 70), that growth weakens. At $T=75$, the wave amplitude starts to decrease. Thereafter, it continues to change but it remains small and tends toward a constant value: the horizontal velocity induced by the instability is of order 10^{-3} at the end of the simulation. Figure 9 shows that the unstable mode stabilizes the mean flow by reducing the gradient of the mean velocity. This mode does not lead to the onset of a turbulent mixed shear layer. The flow is supercritically stable.

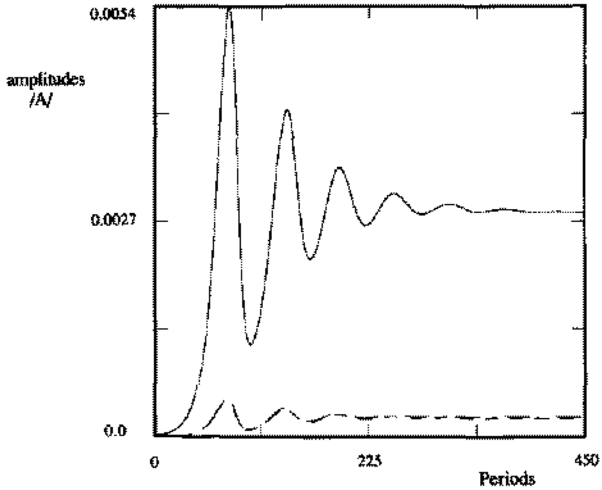

Figure 8 Nonlinear growth of an instability due to large Newtownian cooling. The parameters are those of Figure 7. Amplitude of the horizontal velocity induced by the wave at different heights: $z=0$ (——); $z=+0.1$ (– –); $z=-0.1$ (- - -).

3. INFLUENCE OF DISSIPATION ON NONLINEAR GRAVITY WAVE CRITICAL LEVEL INTERACTION

As mentioned in the introduction, the reflection and transmission of a gravity wave strongly depends upon the value of the Richardson number at the critical level. In turn, the wave induced mean flow stability change at this level is an important feature of the nonlinear dynamics.

3.a Rayleigh friction and Newtownian cooling ($Pr=1$)

To gain physical insight into the nonlinear modification of the mean flow, let us initially consider a gravity wave interacting with a shear layer when dissipation is represented by Rayleigh friction and Newtownian cooling. In this case, the assumption that the critical level is controlled by dissipation rather than by the unsteadiness implies that $l_t \ll a'=a/k$. When the Prandtl number $Pr=a/b=1$, the equations which drive the mean flow changes induced by the wave are

$$a\bar{u} \approx -\frac{1}{2}\frac{d}{dz}[\mathrm{Re}(w_1^* u_1)], \tag{3.1}$$

$$a\bar{\varphi} \approx -\frac{1}{2}\frac{d}{dz}[\mathrm{Re}(w_1^* \varphi_1)]. \tag{3.2}$$

For small disturbances, the forcing terms in these equations can be estimated using

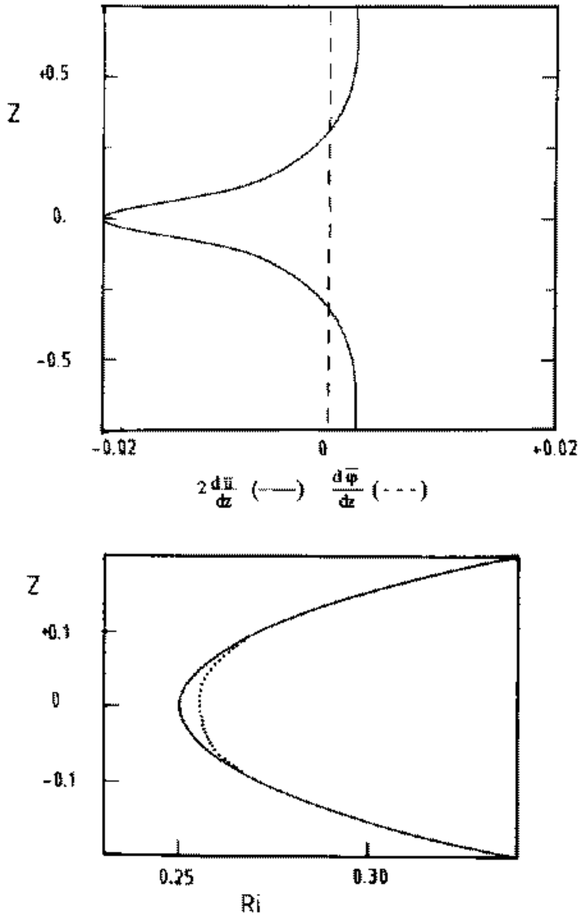

Figure 9 Nonlinear deformation of the mean shears as a function of z when a nonlinear stationary solution is reached. The parameters are those of Figure 7: (a) mean shears; (b) mean Richardson number: initial (____); final (.....).

the linear structure of the wave. The governing equations are

$$(a+ikU)\Delta\psi_1+2ikU(1-U^2)\psi_1+Jik\varphi_1=0, \tag{3.3}$$

$$(a+ikU)\varphi_1-ik\psi_1=0. \tag{3.4}$$

Far from the critical level, $z \gg a'$ (the outer region), the solution can be approximated to first order in a' expanding the solution as a series

$$\psi_1=\psi_1^1+a'\psi_1^2+\cdots; \qquad \varphi_1=\varphi_1^1+a'\varphi_1^2+\cdots,$$

and neglecting the terms of order a' in (3.3) and (3.4). Then, ψ_1^1 is given by the Taylor

Goldstein equation

$$L(\psi_1^1)=\Delta\psi_1^1+\left[\frac{J}{U^2}+2(1-U^2)\right]\psi_1^1=0.$$

Van Duin and Kelder (1982) obtained solutions to this equation for $k^2<J$ (i.e., propagating gravity waves). They reported that the solution on each side of the critical level can be written in terms of hypergeometric functions. Approaching the critical level (i.e. $z\to 0$), this outer solution is

$$\psi_1^1(z)\approx A^{\pm}|z|^{0.5+i\sigma}+B^{\pm}|z|^{0.5-i\sigma}, \qquad \text{where} \quad \sigma=\sqrt{J-0.25} \qquad \text{for} \quad J>0.25,$$

$$\psi_1^1(z)\approx A^{\pm}|z|^{0.5+\sigma'}+B^{\pm}|z|^{0.5-\sigma'}, \qquad \text{where} \quad \sigma'=\sqrt{0.25-J} \qquad \text{for} \quad J>0.25,$$

where the values of the ratios A^+/A^- and B^+/B^- are determined connecting the upper and the lower half plane through the critical layer.

For $z\approx O(a')$, the advection terms in (3.3) and (3.4) are balanced by the damping terms making the preceding development invalid. The equations have to be solved in the inner layer. To first order in a', the unstable mode satisfies the inner equations

$$(y-i)^2\frac{\partial^2\psi_1^I}{\partial y^2}+J\psi_1^I=0; \qquad \varphi_1^I=\frac{1}{a'}\frac{\psi_1^I}{y-i}, \tag{3.5}$$

where $z=a'y$.

The solution of (3.5) is

$$\psi_1^I=A(y-i)^{0.5+i\sigma}+B(y-i)^{0.5-i\sigma} \qquad \text{for} \quad J>0.25, \tag{3.6}$$

$$\psi_1^I=A(y-i)^{0.5+\sigma'}+B(y-i)^{0.5-\sigma'} \qquad \text{for} \quad J<0.25. \tag{3.7}$$

Expanding this solution for $y\to\pm\infty$, it recovers the outer solution expanded for $z\to 0$ if

$$A=a'^{0.5+i\sigma}A^+, \quad B=a'^{0.5-i\sigma}B^+, \quad A^-=-i\exp(\sigma\pi)A^+, \quad B^-=-i\exp(-\sigma\pi)B^+,$$

for $J>0.25$, and

$$A=a'^{0.5+\sigma'}A^+, \quad B=a'^{0.5-\sigma'}B^+, \quad A^-=-i\exp(i\sigma'\pi)A^+, \quad B^-=-i\exp(-i\sigma'\pi)B^+,$$

for $J<0.25$. These last relations are similar to those obtained when an analytic continuation of the solution is performed through the critical level (Miles, 1961) in the inviscid case. Then, the outer solution, corresponds to the inviscid solution determined analytically by Van Duin and Kelder (1982): A^+ and B^+ are known and can be expressed in term of Γ-functions. The forcing terms in (3.1) and (3.2) at the order considered herein, are zero away from the inner layer since the outer solution is steady and undamped. Consequently, these forcing terms are mostly important within the inner layer. They can be calculated analytically from expressions (3.6) and (3.7) because

$$-\frac{1}{2}\mathrm{Re}\frac{d(w_1^* u_1)}{dz} \approx -\frac{1}{2a'^2}\mathrm{Re}\left[ik\frac{d}{dy}\left(\psi_1^{I*}\frac{d\psi_1^I}{dy}\right)\right],$$

$$-\frac{1}{2}\mathrm{Re}\frac{d(w_1^* \varphi_1)}{dz} \approx -\frac{1}{2a'^2}\mathrm{Re}\left[ik\frac{d}{dy}\left(\psi_1^{I*}\frac{d\psi_1^I}{dy}\right)\right],$$

They are shown in the Figure 10 for

$$J=0.23,\ 0.5 \text{ and } 2.5;\ a'=a/k=0.1,\ k=0.1 \text{ and } \varepsilon=1.$$

The mean flow distortion induced by these forcing terms can be described as follows; For all values of J, the wave does not give horizontal momentum to the mean flow at the critical level

$$\bar{u}(z=0)=0,$$

so the critical level does not move. For small J ($J=0.23$ Figure 10a), the linear reflection and transmission of the wave are of order 0.5 and the action of the wave on the mean flow is important on both sides of the critical level. The wave mostly tends to take horizontal momentum and potential temperature from the mean flow above the critical level and to deposit them below. Consequently, both the mean stratification and velocity shear decrease at the critical level. As is true for Kelvin-Helmholtz instabilities, the wave has two opposite effects on the mean flow stability. As long as $J<0.25$, the destabilization at the critical level dominates. For $J>0.25$ (Figure 10b, $J=0.5$), the mean flow is stabilized at the critical level. These results can be summarized as follows:

$$\text{For } J\begin{cases}>0.25\\<0.25\end{cases},\quad (\delta Ri)_{z\approx 0}\approx J\left(\frac{d\bar{\varphi}}{dz}-2\frac{d\bar{u}}{dz}\right)_{z\approx 0}\begin{cases}>0 & \text{mean flow stabilization}\\<0 & \text{mean flow destabilization.}\end{cases}$$

For larger J (Figure 10c, $J=2.5$), the wave absorption is large at the critical level and the wave mean flow interaction only occurs below it. The wave absorption essentially induces an horizontal momentum flux convergence which is very large as compared with the thermal forcing. In this case, the changes occurring at the critical level are negligible when compared with those occurring below.

Furthermore, we found that the wave takes kinetic energy from the mean flow whatever the value of J is

$$C(\bar{K},K')>0.$$

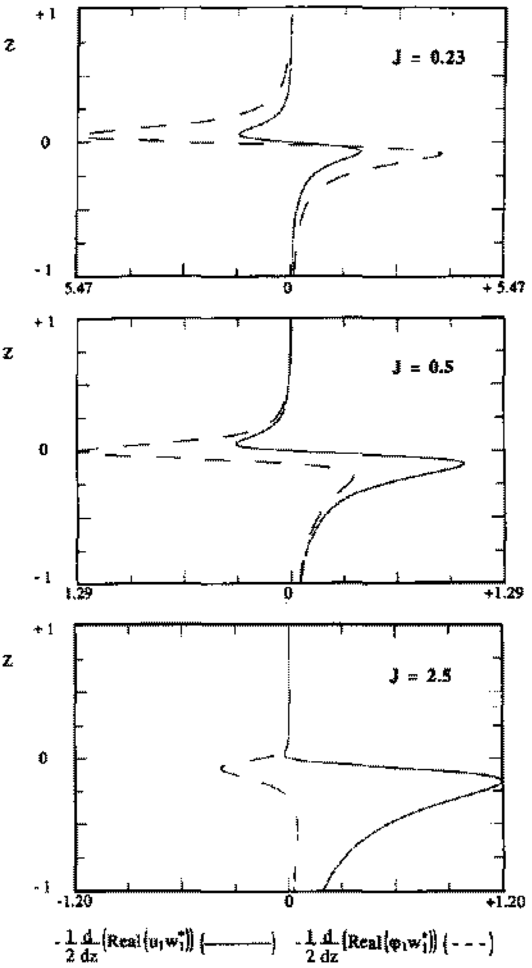

Figure 10 Action of a gravity wave on the mean flow as a function of the height. The unchanged parameters are: $k=0.1$, $a=b=0.1$, $\varepsilon=1$, and for various J.

As is true for Kelvin-Helmholtz instabilities, part of this energy is returned to the mean flow as potential energy

$$C(\bar{P}, K')<0.$$

For low Richardson number, this is done through the downward transfer of horizontal momentum and potential temperature which decreases the mean wind shear and stratification at the critical level. These behaviours are similar to those of the unstable Kelvin Helmholtz modes. This suggests that for the disturbances we consider, the mechanisms which drive the Kelvin-Helmholtz instabilities are similar to those driving the gravity wave critical level interaction at low Richardson number. This qualitatively fits in with the mechanistic descriptions of these modes given by Lindzen (1988) and Lindzen and Rosenthal (1983).

In the remainder of this section, we consider viscosity and thermal conduction as dissipative parameters. In this case, we find that most of the results concerning the action of the wave on the mean flow are similar to those obtained with Newtownian cooling and Rayleigh friction. However, we find that the wave gives momentum to the mean flow at the critical level and moves it downward. This effect is small compared to the other effects described above for small J. For large J, this displacement of the critical level becomes a significant feature. Furthermore, when diffusion processes are considered, the distortion of the mean flow does not exactly correspond to the action of the wave on the flow. Nevertheless, for $Pr=1$, the wave induced mean velocity shear and mean stratification diffuse similarly and remain balanced as described above.

3.b Viscosity and thermal conduction $Pr \neq 1$

In Figure 11, the action of the wave on the mean flow is displayed as a function of height for $J=0.25$, $k=0.1$, $l_v=0.1$ and $Pr=0.2, 1, 5$. We find that the action of the wave on the mean flow is close to the one previously discussed. As illustrated on the figure, thermal and dynamic exchanges occur on layers with half depths of approximately $5l_{th}$ and $5l_v$. Further, near the critical level, a decrease in the amplitude of one dissipation coefficient decreases the negative slope of the associated forcing. Consequently, at $J=0.25$, we find

$$\text{For} \quad Pr\begin{Bmatrix} <1 \\ >1 \end{Bmatrix}, \quad -2\left(\frac{d^2}{dz^2}\left[\mathrm{Re}\left(\frac{u_1 w_1^*}{2}\right)\right]\right)_{z\approx 0} \begin{Bmatrix} < \\ > \end{Bmatrix} -\left(\frac{d^2}{dz^2}\left[\mathrm{Re}\left(\frac{\varphi_1 w_1^*}{2}\right)\right]\right)_{z\approx 0}.$$

The diffusion of the induced mean flow changes may further reinforce these effects, since a decrease in the amplitude of one dissipation coefficient decreases the diffusion of the associated mean flow modification. As for Kelvin-Helmholtz instabilities, we assume that the mean flow stability change at the critical level for $J=0.25$ is given by

$$\text{For} \quad Pr\begin{Bmatrix} <1 \\ >1 \end{Bmatrix}, \quad (\delta Ri)_{z\approx 0} \approx J\left(\frac{d\bar{\varphi}}{dz} - 2\frac{d\bar{u}}{dz}\right)_{z\approx 0} \begin{cases} >0 & \text{mean flow stabilization} \\ <0 & \text{mean flow destabilization} \end{cases}.$$

For other values of the minimum Richardson number J, such a sensitivity to the Prandtl number is also observed. Nevertheless, the Prandtl number at which the thermal and the momentum forcing compensates (regarding the mean flow stability change at the critical level), increases when J increases. This relation is shown on Figure 12.

3.c Nonlinear simulations

The former analysis indicates that the nonlinear evolution of the interaction between a gravity wave and a critical level strongly depends on the values of J and Pr. Now

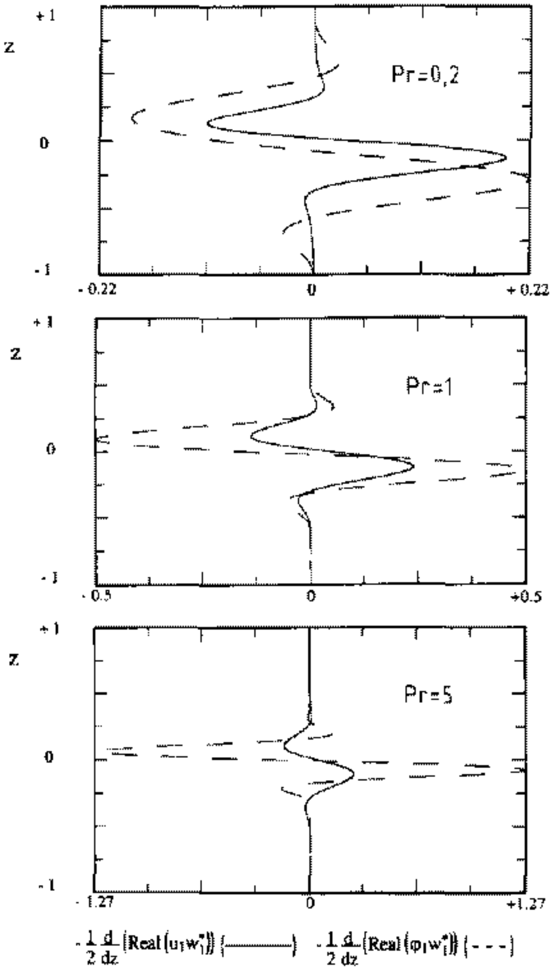

Figure 11 Action of the wave on the mean flow as a function of the height. The unchanged parameters are: $J=0.25$, $l_v=0.1$, $k=0.1$, $\varepsilon=1$, and for various Pr.

we shall present the temporal evolution of the system calculated using the nonlinear time dependant model. In all the following experiments, the horizontal wavenumber and the viscous scale remain unchanged:

$$k=0.1, \qquad l_v=(k\,Re)^{-1/3}=0.1.$$

The function $f(t)$ which drives the amplitude of the wave source (I.3) is given by

$$f(t)=\begin{cases} f_0 \sin\left(\dfrac{\pi}{2}\dfrac{t}{10T}\right) & \text{for } t<10T, \\ f_0 & \text{for } t>10T, \end{cases}$$

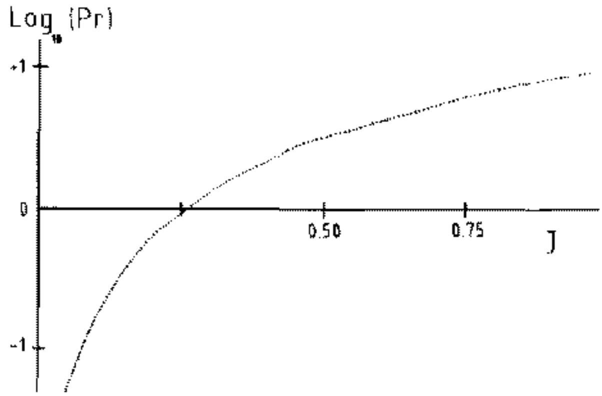

Figure 12 Variation of the Prandtl number versus J for which the action of the wave on the stability of the mean flow at the critical level is neutral:

$$-2\left[\frac{d^2}{dz^2}\left\{\mathrm{Re}\left(\frac{u_1 w_1^*}{2}\right)\right\}\right]_{z=0} \approx -\left[\frac{d^2}{dz^2}\left\{\mathrm{Re}\left(\frac{\varphi_1 w_1^*}{2}\right)\right\}\right]_{z=0}$$

where $T=2\pi/k$ is the Doppler shifted period of the wave far away the shear layer and f_0 allows the tuning of the amplitude, ε, of the incident wave. For the duration of the transient part of the forcing, we verify that $l_t \approx 0.03 \ll l_v$: the critical layer is always controlled by dissipation. Note, a faster transience does not change the results significantly (at least at large t). Indeed, after a long time, the amplitude of the wave source is constant and the temporal variation of the fundamental mode remains slow enough so that the interaction is controlled by dissipation. In subsequent paragraphs, we are going to present results for three particular values of J: 0.23, 0.5 and 2.5. This allows a qualitatively complete description of the phenomena observed in many simulations with various J. In all the cases studied, the nonlinear effects are significant when the nonlinear scale is of the order of magnitude of the smallest viscous scale

$$l_n \approx \mathrm{Min}(l_v, l_{th}).$$

This relation is satisfied in all the simulations to come (unless the wave amplitude is explicitly given). Furthermore, we find that the critical level interaction is convectively unstable when $l_n > 2\,\mathrm{Min}(l_v, l_{th})$.

In the numerical model, the amplitude of the incident wave, ε, the reflection and transmission coefficients $|R|$, and $|T|$ are deduced from the structure of the fundamental mode. Far away the shear layer, this mode is not affected by dissipation and can be written as

$$\psi_1^1 \approx \varepsilon(e^{imz} + R\,e^{-imz})$$

below the critical level, and as

$$\psi_1^1 \approx \varepsilon T e^{-imz}$$

above it (here, $m^2 = J - k^2 > 0$).

3.c.1) Low initial minimum Richardson number: $J=0.23$; $Pr=0.2, 1, 5$

Figure 13 displays the temporal evolution of the reflection and transmission coefficients of the gravity wave in comparison with linear model results. Note first that at $T=20$, these coefficients increase when the Prandtl number increases. To a certain extent, these features can be related to the mean flow changes illustrated in Figure 14 at $T=20$. In all cases, the critical level does not move down significantly and we can consider that it remains located a $z=0$. At the critical level, in agreement with the aforementioned calculations, the mean flow is stabilized when $Pr=0.2$, weakly destabilized when $Pr=1$ and destabilized when $Pr=5$. However, the nonlinear values of $|R|$ and $|T|$ are also due to modifications of the background flow occurring below the critical level. For instance, at $Pr=1$, the nonlinear reflection coefficient is significantly larger (25%) than the linear one while the nonlinear transmission coefficient is smaller. None of these results can be related to the very small change of the mean flow stability at the critical level. In fact, as noted previously, the mean flow distortions are also important below the critical level. For $Pr=1$, Figure 14 shows that at $T=20$, the minimum value of the Richardson number is 0.2 and is

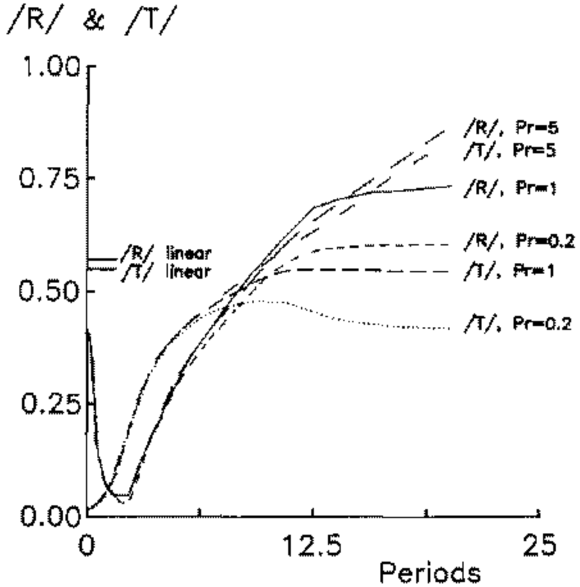

Figure 13 Temporal evolution of the reflection and transmission coefficients of a gravity wave propagating toward a critical level for various Prandtl numbers. The unchanged parameters are: $J=0.23$, $k=0.1$, $l_0=0.1$.

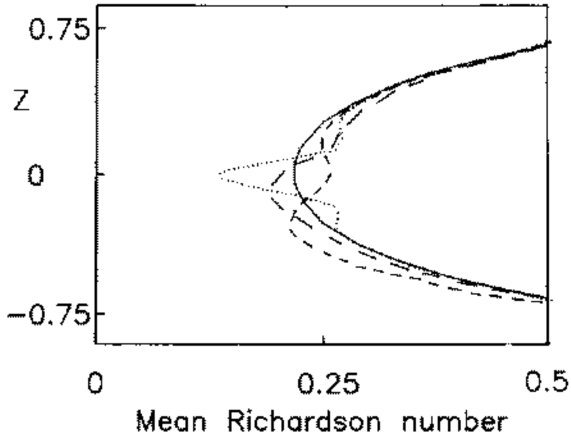

Figure 14 Nonlinear changes in the mean flow stability at $t=20\times T$. The parameters are those of Figure 13: initial (——); $Pr=0.2$ (- - -), $Pr=1$ (— —), $Pr=5$ (.....)

located below the critical level. This destabilization is related to the wind shear decrease occurring below the maximum of horizontal momentum flux convergence. This distortion induces partial reflection of the incident wave before it reaches its critical level. Thus, the reflection coefficient is larger than that of the linear model. The transmission coefficient is lower because less wave energy reaches the critical level. At small Prandtl number ($Pr=0.2$), the stabilization of the mean flow occurring at the critical level renders the reflection and transmission coefficients smaller than at $Pr=1$. At large Prandtl number ($Pr=5$), mean flow destabilization at the critical level is large enough to make both the reflection and the transmission of the wave greater than was true for the linear case. Furthermore, in this case, the mean flow modifications occurring below the critical level are less important than at $Pr=1$ and 0.2. Indeed, to maintain the condition that $l_n \approx \mathrm{Min}(l_v, l_{th})$, it is necessary to decrease the amplitude, ε, of the wave. Thus, there is less momentum deposition below the critical level and partial reflection is smaller than it was for $Pr \leqslant 1$. Also for $Pr=5$, the reflection and transmission coefficients are quantitatively similar and we presume that they are largely associated with the value of the mean Richardson number at the critical level.

In these experiments, the wave is forced by a source but unstable modes can also grow in the flow. The growth rate of the most unstable mode is of order $\mu=0.25-J$. Thus, the time scale characterizing the growth of this mode, $1/\mu \approx 50$ is close to one period of the forced wave: $2\mu/k \approx 63$. Furthermore, the amplitude of the wave source varies slowly as compared with one period of the wave ($l_t \ll 1$). Consequently, unstable modes increase very rapidly. Nevertheless, the importance of these modes is also determined by their amplitudes at the initial time. In the experiments presented herein, the unstable modes are controlled by the amplitude of the wave through nonlinear interaction. At $Pr=0.2$, since the wave stabilizes the mean flow at the shear layer, the growth of the unstable modes is stopped before they reach significant values. The forced wave is the main disturbance present at the critical level as long as it is imposed by the wave source. At $Pr = 1$, the disturbance is dominated by the forced wave during more than 20 periods. Thereafter, unstable modes dominate the gravity wave and lead to convective overturning. For $Pr = 5$, the wave destabilizes the mean flow, and the unstable modes grow more rapidly than at $Pr = 1$. At $t = 20T$, the unstable modes dominate the forced wave at the shear layer. Thereafter, since these modes also destabilize the flow, (as found in the Section 2), they continue to grow and lead to convective overturning. In that case, the presence of the gravity wave accelerates the onset of turbulence in the shear layer.

Other experiments, conducted with smaller J, give similar results. We verified that the value of the Prandtl number, for which the mean flow stability at the critical level doesn't change, decreases when J decreases. Furthermore, the induced nonlinear modification of the reflection and transmission of the wave follows the stability modifications as described above. However, at very small J (e.g., $J = 0.1$) the unstable modes grow so rapidly that they dominate the disturbance after few periods of the forced wave. Then, the critical layer is unsteady and rapidly becomes unstable.

3.c.2) *Initial minimum Richardson number: $J = 0.5$; $Pr = 1, 10$*

The results obtained with $J = 0.5$ are close to those obtained at $J = 0.23$ except that at $Pr = 1$, the mean flow is stabilized at the critical level (Figure 16). The induced nonlinear reflection coefficient of the incident wave is larger than the linear one (Figure 15). The transmission coefficient is smaller. These features are related to the partial reflection occurring below the critical level. Note that, as discussed in (3.c.3), it increases when the wave amplitude increases. It is necessary to increase the Prandtl number up to $Pr = 10$ to find mean flow destabilization. Then, as observed on Figure 16, both the reflection and transmission of the wave are larger than they were in the linear case and are comparable. At larger l_n, these effects are more

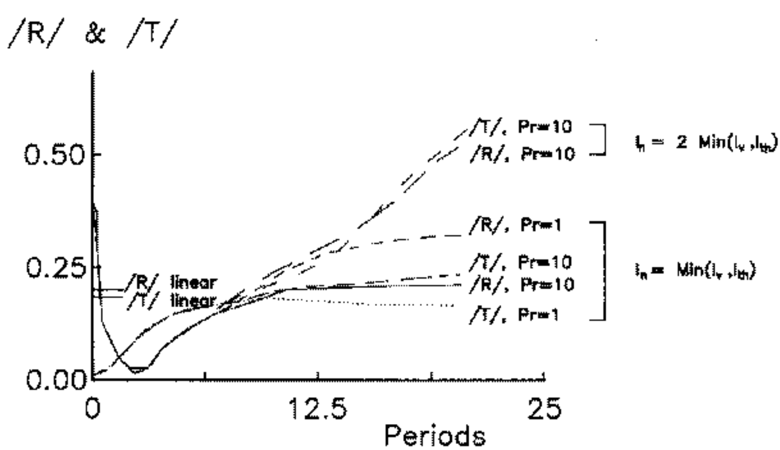

Figure 15 Temporal evolution of the reflection and transmission coefficients of a gravity wave propagating toward a critical level for various Prandtl numbers and wave amplitudes. The unchanged parameters are: $J = 0.5$, $k = 0.1$, $l_v = 0.1$.

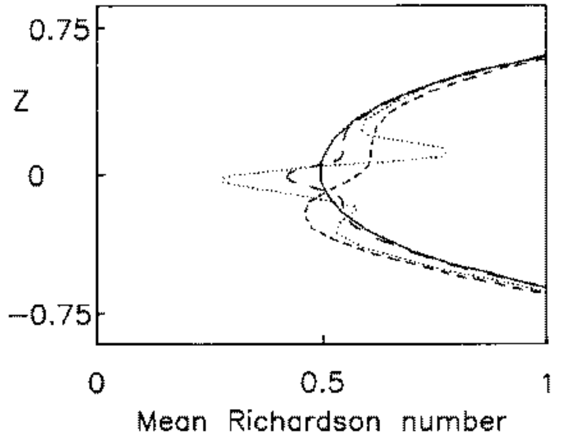

Figure 16 Nonlinear changes in the mean flow stability at $t=20\times T$. The parameters are those of Figure 15: initial (——); $Pr=1$, $\ln=\mathrm{Min}\ (l_v, l_{th})$ (- - -); $Pr=10$, $\ln=\mathrm{Min}\ (l_v, l_{th})$ (— —), $Pr=10$, $\ln=2\ \mathrm{Min}\ (l_v, l_{th})$ (.....).

pronounced. For instance, at $l_n \approx 2\,\mathrm{Min}(l_v, l_{th})$ and $Pr=1.0$, the mean Richardson number at the critical level decreases to 0.25. In this case, the reflection and transmission coefficients are comparable and exceed 0.5. This means that the quantity $|R|+|T|$ is larger than one. This feature corresponds to the onset of over-reflection when the transmitted wave is not allowed to propagate far above the shear layer (Lindzen and Barker, 1985). In the linear case, such a situation only occurs for unstable flows (i.e. for $J<0.25$). In initially stable flow, the occurrence of large nonlinear reflection and small nonlinear transmission when $Pr=1$, does not characterize the nonlinear dissipative regime. It was early found by Brown and Stewartson (1982)a, b) in the nonlinear inviscid case. Nevertheless, the occurrence of large transmission and $|R|+|T|>1$ at large Prandtl number is a new result. Furthermore, in these simulations, the secondary modes, which can be local instabilities of the flow (Fritts, 1978) remain small. They become important for larger wave amplitude. In that case, they precede convective overturning.

3.c.3) Initial minimum Richardson number $J=2.5$; $Pr=1$

For large Richardson number and $Pr=1$, the wave does not interact with the mean flow above the critical level. Furthermore, the modification of the mean flow stratification by the wave is small as compared to the modification of the mean shear. This renders a discussion of the sensitivity to the Prandtl number beyond the scope of this study: mean flow destabilization only occurs at very large Prandtl number. At $Pr=1$, the partial reflection occurring below the critical level is the most important nonlinear effect. Consequently, the total nonlinear reflection of the incident wave can be large while the transmission always remains very small. On one hand, this is the direct opposite of the result of Breeding (1971), who claimed that at large Richardson number the nonlinear critical level remains the wave absorber predicted by the linear theory. We presume that the numerical resolution used by this author near the critical level was not sufficiently accurate to resolve the interaction correctly. On the other hand, it carries over into the dissipative regime, the nonlinear inviscid results of Brown and Stewartson (1982a, b) aforementioned. The nonlinear effects become more and more important when the wave amplitude is increased. At the above limit when the secondary modes become important and convective overturning occurs (approximately when $l_n \approx 2l_v$), the total reflection is large. Furthermore, when maintaining the ratio $l_n/l_v \approx 2$, but decreasing the viscous scale, l_v, toward 0, the nonlinear reflection approaches 1 (the value $|R| = 0.75$ has been actually reached). This increase in the nonlinear reflection is simply due to the fact that at small l_v, the mean flow changes are more abrupt below the critical level making partial reflections there more and more important.

CONCLUSIONS

The present analysis of the nonlinear interaction between a disturbance and its critical level have revealed several interesting results concerning the weakly nonlinear dissipative regime. This regime was defined numerically by the relation: $l_n \approx \mathrm{Min}(l_v, l_{th}) \gg l_t$ for gravity waves and by $l_n \approx l_v \gg l_t$ for Kelvin-Helmholtz instabilities. The impact of secondary modes on the disturbance is small for these scale ranges as compared to the nonlinear mean flow distortion. For smaller values of l_n, Lott and Teitelbaum (1990) show that the linear approximation is valid. For larger l_n [i.e. $l_n > 2\,\mathrm{Min}(l_v, l_{th})$], secondary modes are large in the shear layer and the system can become convectively unstable. These scale relations roughly represent what was observed experimentally in the simulations presented.

We show that weakly unstable Kelvin-Helmholtz waves grow and compensate their energy loss due to dissipation removing kinetic energy from the mean flow. Furthermore, part of this energy is restored to the mean flow through potential energy. These exchanges induce a decrease of the mean wind shear and the mean stratification at the critical level. As a consequence, they determine the changes in the stability conditions. When dissipation controls the interaction, the nonlinear stabilization of the mean flow at low Prandtl number ($Pr < 1$) induces a stabilization of the wave whereas at large Prandtl number ($Pr > 1$) the opposite is true. We examined these effects in detail. Thermal dissipation plays a destabilizing role, since induced thermal exchanges decrease the mean flow stratification around the critical level. Conversely, viscosity plays a stabilizing role by decreasing the mean wind shear. Thus, it is surprising that destabilization occurs for $Pr > 1$. In fact, when the thermal conduction is small, the thermal exchanges occur in a thin layer. This induces a sharp and negative slope of mean potential temperature modification which is only weakly damped because thermal conduction is small. The inverse of this analysis is also valid for viscosity. Then, the stability of the mean flow is controlled by the Prandtl number through the lengths l_{th} and l_v associated to thermal conduction and viscosity.

For small Prandtl number, non-linearities stabilize the flow, contrary to the linear case where unstable modes can occur even for $J > 0.25$. In this case, since instabilities are due to dissipation, the latter necessarily dominate the unsteadiness during the interaction. It is found that the growth of such a mode is rapidly limited by nonlinear effects: it is supercritically stable.

Similar properties are found when the disturbance is a gravity wave forced by a source located far below the critical layer. The condition for which the interaction is dissipative rather than unsteady is linked to the temporal variation of the wave source. Furthermore, in all the cases studied, the interaction necessarily becomes dissipative after the wave source has reached a constant value (when no unstable mode is simultaneously present). For stable as well as unstable flow, a decrease of the Prandtl number has a nonlinear stabilizing effect on the mean flow and on the disturbance. The value of the Prandtl number for which destabilization of the mean flow at the critical level (through a decrease of the mean stratification) compensates its stabilization (through a decrease of the mean wind shear) increases when the minimum Richardson number J increases. For $J=0.25$, this "critical value" of the Prandtl number is $Pr=1$.

For weakly unstable flow ($J=0.23$), the critical value of the Prandtl number is close to one. Then, as is true for Kelvin-Helmholtz instabilities, a gravity wave can stabilize (destabilize) the mean flow at the critical level when $Pr<1$ ($Pr>1$). When the mean flow is stabilized, the unstable modes which initially grow are damped and the forced wave remains the dominant disturbance at the critical level. In this case, the nonlinear reflection of the wave is larger than in the linear case and its transmission is smaller. These features are due to the combination of partial reflection occurring below the critical level and of mean flow stabilization occurring at the critical level. When the wave destabilizes the mean flow, the unstable modes grow faster than in the linear case and rapidly dominate the forced wave at the critical layer. Nevertheless, during the first step of the evolution, both the nonlinear reflection and transmission of the wave are comparable to one other and are larger than in the linear case. At $J=0.5$, the mean flow distortion has similar effects on the reflection and transmission of the wave. At $Pr=1$, the mean flow is stabilized at the critical level and the nonlinear reflection of the wave is larger than in the linear case, its transmission is smaller than in the linear case. At $Pr=10$, the mean flow is destabilized at the critical level, both the reflection and transmission of the wave are comparable and larger than in the linear case. Then, for large amplitude waves [i.e. such that $l_n \approx 2\,\mathrm{Min}(l_v, l_{th}]$, a situation for which $|R|+|T| \geqslant 1$ was reached because the mean Richardson number becomes smaller than 0.25 at the critical level. For large minimum Richardson number ($J=2.5$) and $Pr=1$, the wave mean flow exchanges mostly occur below the critical level. This renders the discussion concerning the sensitivity of the critical level interaction to the Prandtl number beyond the scope of this study: mean flow destabilization occurs at very large Prandtl number. For $Pr=1$, the induced partial reflection of the incident wave can be large. It is found that the nonlinear reflection coefficient can approach 1 for "large" amplitude waves (i.e. such that $l_n \approx 2l_v$), when dissipation approaches 0 (i.e. $l_v -> 0$). The transmission remains very small.

Acknowledgements

The authors are pleased to acknowledge some very helpful suggestions of Dr. Maura Hagan of M.I.T., Haystack observatory, and Dr. François Vial of Laboratoire de Météorologie Dynamique. The computations presented in this paper were funded by the Scientific Council of the Centre de Calcul Vectoriel pour la Recherche.

References

Booker, J. R. and Bretherton, F. P., "The critical layer for internal gravity waves in a shear flow," *J. Fluid Mech.* **25,** 513–529 (1967).

Breeding, R. J., "A non linear investigation of critical levels for internal atmospheric gravity waves," *J. Fluid Mech.* **50,** 545–563 (1971).

Brown, S. N. and Stewartson, K., "The evolution of a small inviscid disturbance to a marginally unstable stratified shear flow; stage 2," *Proc. R. Soc. Lond.* **A363,** 175–194 (1978).

Brown, S. N. and Stewartson, K., "On the nonlinear reflection of a gravity wave at a critical level, Part 1," *J. Fluid Mech.* **100,** 577–595 (1980).

Brown, S. N. and Stewartson, K., "On the nonlinear reflection of a gravity wave at a critical level, Part 2," *J. Fluid Mech.* **115,** 217–230 (1982a).

Brown, S. N. and Stewartson, K., "On the nonlinear reflection of a gravity wave at a critical level, Part 3," *J. Fluid Mech.* **115,** 231–250 (1982b).

Brown, S. N., Rosen, A. S. and Maslowe, S. A., "The evolution of a quasi-steady critical layer in a stratified viscous shear flow," *Proc. R. Soc. Lond.* A. **375,** 271–293 (1981).

Churilov, S. M. and Shukhman, I. G., "Non linear stability of a stratified shear flow: a viscous critical layer," *J. Fluid Mech.* **180,** 1–20 (1987).

Churilov, S. M. and Shukhman, I. G., "Nonlinear stability of a stratified shear flow in the regime with an unsteady critical layer," *J. Fluid Mech.* **194,** 187–216 (1988).

Clark, T. L. and Peltier, W. R., "Critical level reflection and the resonant growth of nonlinear mountain waves," *J. Atmos. Sci.* **41,** 3122–3134 (1984).

Collins, D. A. and Maslowe, S. A., "Vortex pairing and resonant wave interactions in a stratified free shear layer," *J. Fluid Mech.* **191,** 465–480 (1988).

Drazin, P. G., "The stability of a shear layer in an unbounded heterogeneous inviscid fluid," *J. Fluid Mech.* **4,** 214–224 (1958).

Fitts, D. C., "The nonlinear gravity wave critical-level interaction," *J. Atmos. Sci.* **35,** 397–413 (1978).

Fritts, D. C., "The transient critical-level interaction in a Boussinesq fluid," *J. Geophys. Res.* **87,** 7997–8016 (1982).

Hazel, P., "The effect of viscosity and heat conduction on internal gravity waves at a critical level," *J. Fluid Mech.* **30,** 775–783 (1967).

Howard, L. N., "Note on a paper of John W. Wiles," *J. Fluid Mech.* **10,** 509–512 (1961).

Howard, L. N., "Neutral curves and stability boundaries in stratified flow," *J. Fluid Mech.* **16,** 333–342 (1963).

Jones, C. A., "The onset of shear instability in stars," *Geophys. Astrophys. Fluid Dynam.* **8,** 165–184 (1977).

Jones, W. L., "Reflection and stability of waves in stably stratified fluid with shear flow: a numerical study," *J. Fluid Mech.* **34,** 609–624 (1968).

Klassen, G. P. and Peltier, W. R., "Evolution of finite amplitude Kelvin Helmholtz Billows in two spatial dimensions," *J. Atmos. Sci.* 1321–1339 (1985a).

Klassen, G. P. and Peltier, W. R., "The effect of the Prandtl number on the evolution and stability of Kelvin Helmholtz billows," *Geophys. Astrophys. Fluid Dynam.* **32,** 23–60 (1985b).

Lindzen, R. S., "Instability of plane parallel shear flow (toward a mechanistic picture of how it works)," *Pageoph.* **126,** no. 1, 103–121 (1988).

Lindzen, R. S. and Barker, J. W. "Instability and wave overreflection in stably stratified shear flow," *J. Fluid Mech.* **151,** 189–217 (1985).

Lindzen, R. S. and Rosenthal, A. J., "Instabilities in a stratified fluid having one critical level, Part III: Kelvin-Helmholtz instabilities as over-reflected waves," *J. Atmos. Sci.* **40,** 530–542 (1983).

Lott, F. and Teitelbaum, H., "Influence of dissipation on gravity waves propagating through a shear layer and on instabilities: validity of the linear approximation," *Annales Geophysicae* **8,** (1), 37–52 (1990).

Maslowe, S. A., "Weakly nonlinear stability theory of stratified shear flows," *Quart. J. R. Met. Soc.* **103,** 769–783 (1977).

Maslowe, S. A., "Critical layer in shear flows," *Annu. Rev. Fluid Mech.* **18**, 405–432 (1986).
Maslowe, S. A. and Thompson, J. M., "Stability of a stratified shear layer," *Phys. Fluids* **14**, 453–458 (1971).
Miles, J. W., "On the stability of heterogeneous shear flows," *J. Fluid Mech.* **10**, 498–508 (1961).
Miller, J. R. and K. S. Gage, "Prandtl number dependance of the stability of stratified free shear layer," *Phys. Fluids* **15**, 723–725 (1972).
Miller, R. L. and Lindzen, R. S., "Viscous destabilization of stratified shear flow for $Ri > 1/4$," *Geophys. Astrophys. Fluid Dynam.* **42**, 49–91 (1988).
Rosenthal, A. J. and Lindzen, R. S., "Instabilities in a stratified fluid having one critical level, Part II: explanation of gravity wave instabilities using the concept of overreflection," *J. Atmos. Sci.* **40**, 521–529 (1983).
Smyth, W. D. and W. R. Peltier, "The transition between Kelvin Helmholtz and Holmboe instability: an investigation of the overreflection hypothesis," *J. Atmos. Sci.* **46**, 3698–3720 (1989).
Stewartson, K., "Marginally stable inviscid flows with critical layers," *I.M.A.J. Appl. Math.* **27**, 133–175 (1981).
Van Duin, C. A. and Kelder, H., "Reflection properties of internal gravity waves incident upon a hyperbolic tangent shear layer," *J. Fluid Mech.* **120**, 505–521 (1982).
Van Duin, C. A. and Kelder, H., "Internal gravity waves in shear flows at large Reynolds number," *J. Fluid Mech.* **169**, 293–306 (1986).

APPENDIX A

The non linear scale characterizes the distance to the critical level at which the non linear terms balance the linear ones. Considering the disturbance as a normal mode $\psi_1(z)\exp(ikx)$, near the critical level we have (see for instance Booker and Bretherton, 1967):

$$\psi_1(z) \approx A^+ z^{0.5+i\sigma} + B^+ z^{0.5-i\sigma} \qquad \text{where} \quad \sigma = \sqrt{J-0.25} \qquad \text{for} \quad J > 0.25,$$

$$\psi_1(z) \approx A^+ z^{0.5+\sigma'} + B^+ z^{0.5-\sigma'} \qquad \text{where} \quad \sigma' = \sqrt{0.25-J} \qquad \text{for} \quad J < 0.25,$$

where ψ_1 is a stream function. Roughly comparing the amplitude of the linear advection term $U(ik\Delta\psi_1)$ to that of the non linear one $\partial_z\psi_1(ik\Delta\psi_1)$ at l_n leads to

$$\frac{k|\psi_1|}{l_n} \approx \frac{k|\psi_1|^2}{l_n^3} \Rightarrow l_n^2 \approx |\psi_1|,$$

for $\quad J > 0.25, \ |\psi_1| \approx \mathrm{Max}(|A^+|, |B^+|) l_n^{1/2} \Rightarrow l_n \approx \mathrm{Max}(|A^+|, |B^+|)^{2/3},$

for $\quad J < 0.25, \ |\psi_1| \approx |B^+| l_n^{1/2-\sigma'} \qquad \Rightarrow l_n \approx |B^+|^{1/(3/2+\sigma')}.$

For given k and J, A^+ and B^+ varies linearly with the amplitude, ε, of the wave,

for $J > 0.25$, l_n varies as $\varepsilon^{2/3}$,
for $J < 0.25$, l_n varies as $\varepsilon^{2/(3+2\sigma')}$.

APPENDIX B: ENERGETIC BUDGET FOR TWO-DIMENSIONAL DISTURBANCES

In the Boussinesq approximation, the equations (1.1) and (1.2) satisfy an energy closure close to that presented by Klassen and Peltier (1985a). It involves the following energy reservoirs:

$$\bar{K}=\tfrac{1}{2}\langle\overline{u_T}^2\rangle, \text{ the mean kinetic energy; } \overline{u_T}=U+\bar{u},$$

$$K'=\tfrac{1}{2}\langle\overline{u'^2}+\overline{w'^2}\rangle, \text{ the wave kinetic energy,}$$

$$\bar{P}=-J\langle z\bar{\varphi}\rangle, \text{ the mean potential energy.}$$

Here, the integral operators are

$$\overline{(\;)}=k/2\pi\int_0^{2\pi/k}(\;)\mathrm{d}x \qquad \text{and} \quad \langle(\;)\rangle=\int_{-H}^{+H}(\;)dz \qquad \text{with} \quad 1\ll H<-z_S.$$

These energy reservoirs represent the average energy per unit horizontal area contained in a fluid column of height $2H$. Assuming that the eddy interaction terms are sufficiently small, third order nonlinear terms can be neglected and the energetic balances can be written as

$$\frac{d\bar{K}}{dt}=-C(\bar{K},K')-D(\bar{K}),$$

$$\frac{dK'}{dt}=+C(\bar{K},K')+C(\bar{P},K')-[F_W]_{-H}^{+H}-D(K'),$$

$$\frac{d\bar{P}}{dt}=-C(\bar{P},K')-D(\bar{P}),$$

where,
$$C(\bar{K},K')=\left\langle\overline{u_T}\frac{\partial(\overline{u'w'})}{\partial z}\right\rangle,$$

$$C(\bar{P},K')=\left\langle -Jz\frac{\partial(\overline{\varphi'w'})}{\partial z}\right\rangle,$$

$$D(\bar{K})=\frac{1}{Re}\left\langle\overline{u_T}\frac{d^2\bar{u}}{dz^2}\right\rangle-a\langle\bar{u}^2\rangle,$$

$$D(\bar{P}) = \frac{J}{Pr\,Re}\left\langle z\frac{d^2\bar{\varphi}}{dz^2}\right\rangle - bJ\langle z\bar{\varphi}\rangle,$$

$$D(K') = \frac{1}{Re}\left\langle \overline{\frac{\partial u'^2}{\partial x}} + \overline{\frac{\partial u'^2}{\partial z}} + \overline{\frac{\partial w'^2}{\partial x}} + \overline{\frac{\partial w'^2}{\partial z}} \right\rangle + a\langle \overline{u'^2} + \overline{w'^2}\rangle - \frac{1}{Re}\left[\overline{u'\frac{\partial u'}{\partial z}} + \overline{w'\frac{\partial w'}{\partial z}}\right]_{-H}^{+H}.$$

Here, the notation $C(\alpha, \beta)$ denotes the conversion of energy from reservoir α to reservoir β, while positive $D(\alpha)$ represents a loss of energy from reservoir α due to diffusion.

$$F_w = \overline{p'w'} + \overline{u_T}\,\overline{u'w'} - Jz\overline{w'\varphi'}$$

is the flux of total energy carried by the wave, $\overline{p'w'}$ represents the total mean work in the system due to the wave, $\overline{u_T}\,\overline{u'w'}$ and $-Jz\overline{w'\varphi'}$ are the vertical fluxes of kinetic and potential energy carried by the wave. For Kelvin-Helmholtz instabilities, F_w is close to zero (for sufficiently large H) since the wave is evanescent away from the shear layer. For gravity waves interacting with a shear layer, this term is also close to zero when dissipation is small and when the system is nearly stationary.