Floquet Theory Applied to the Analysis of Climate Feedbacks' Seasonal Cycle;

Determining the Evanescent Floquet Vectors

Krista Reimer

Supervisor: Alain Lahellec

Laboratoire de Météorologie Dynamique, Paris VI, Paris, France

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Preface

The following report outlines the results from the work done during my research internship under the supervision of Alain Lahellec at the Laboratoire de Météorologie Dynamique, Université Pierre et Marie Curie (Paris VI) in Paris, France. The internship started with an introduction to fundamental climate concepts as well as the current state of climate research. Following these preliminaries I familiarized myself with the concept of feedbacks in climate studies and the formal analysis methods already developed by Alain Lahellec and Jean-Louis Dufresne on the topic. The remainder of the project involved work on the application of Floquet theory to the analysis of the seasonal cycle of climate feedbacks. In particular, the work was focused on developing a new method for the numerical determination of Floquet vectors previously indeterminable with the standard methods. Theoretical solutions were tested and implemented with the modeling tool Mini_ker.

1 Introduction

Climate change and greenhouse gas emissions are common topics in political and social debate. While questions loom about what society, industries, and governments should be doing regarding the issue, climatologists are working to continue increasing understanding about the climate and the effects of human activity on it. It is not surprising that the massive yet intricate climate system has demanded in their work the creation of similarly complex models in order to anticipate and imitate its behavior. These models, termed global circulation models (GCM's) within the climate research community, represent through mathematical systems physical processes in the ocean, on land surfaces and in the atmosphere[2]. Their numerical implementation is done using a discretisation to a three dimensional grid over the globe. Within the larger goals of understanding the climate and predicting climate change, one possible aspect of the models is their use in analysing climate feedbacks. Climate feedbacks are the results of interactive climate mechanisms in which the reaction of an initial perturbed mechanism prompts a secondary process which in turn effects, either amplifying or diminishing, the changes due to the initial perturbation. Long-term climate feedbacks are analysed using variables' annual averages, however, the amplitude variation experienced by the climate at equilibrium due to the seasonal solar cycle surpasses the moderate forcings of many feedback experiments[4]. Therefore, important aspects of climate feedbacks may require a look not only at annual averages but also at the effect of the seasonal cycle. Instead of using GCM's to perform the analysis the decision was made to use a perfectly periodic simple climate model in which all the needed operators could be calculated and where Floquet theory can be used to inform the analysis. Already Lahellec has advanced a formal analysis of climate feedbacks developed by Lahellec et al.(see [3], hereafter LAH:I) to incorporate the use of Floquet theory in analysing the effect of the seasonal cycle (see [4], hereafter LAH:II). Numerical difficulties arise, however, in finding the final and evanescent Floquet vectors; vectors which are important for the comprehensive analysis. A new method for their determination is described here along with results from the method applied to the simple climate model. This method has partial success by increasing the number of numerically tractable Floquet vectors from the previous method but failed to provide a means for determining all of the Floquet vectors in the four dimensional simple model. The simple climate model, formal feedback analysis, Floquet Theory and the resulting extension to the feedback analysis already developed by Lahellec et al. are presented briefly below as a background to the new method for determining evanescent Floquet vectors.

2 Background

2.1 Simple Climate Model

The perfectly periodic simple model used to determine the seasonal cycle's effect was developed by Lahellec from the model ClimSI used in Hallegatte et al.[1]. By Lahellec it has been re-written in Mini_ker[1], a solver which symbolically determines all partial derivatives and thus the Jacobian matrices. It is a single atmospheric column model with four state variables; the temperature of a 50 m mixed ocean-layer, the temperatures of both a tropospheric layer and stratospheric layer with constant lapse rates and the tropospheric water-content. The radiation equation, however, is integrated on a 60 layer grid for the two major greenhouse gases of the atmosphere (H2O and CO2) using a Malkmus narrow-band model. The model is stable and has no intrinsic oscillatory component, thus all periodic behavior is due to a forcing which represents the incoming seasonally varying solar flux. A detailed description of the model, which is called Mini-Clim, can be found in Appendix A of LAH:II.

2.2 Formal Feedback Analysis

The following short summary of the formal feedback analysis derived in LAH:I explains how a feedback system is derived from a simple atmospheric state-space model such as the one described above.
Letting $\boldsymbol{\eta}$ be the corresponding n-dimensional state vector and $\boldsymbol{F}_x(t)$ an external forcing, consider the general simple climate state-space model

$$\partial_t \boldsymbol{\eta} = \boldsymbol{g}(\boldsymbol{\eta}, \boldsymbol{F}_x(t)) \; .$$

Once initial conditions $\eta(t=0) = \eta_0$ are known a particular reference trajectory may be found. The process of linearizing the system around such a trajectory, selecting a test variable with which to measure the feedback and decomposing the system's response to perturbation into both a base response and an effective response produces the following effective response system,

$$\begin{cases} \partial_t \boldsymbol{\Delta\eta}^r(t) = M^\flat(t)\, \boldsymbol{\Delta\eta}^r(t) + |\, b(t) > (\Delta\theta + \Delta\theta_0(t)) \\ \Delta\theta(t) = < c \,|\; \boldsymbol{\Delta\eta}^r(t) \; , \end{cases} \tag{1}$$

where $\boldsymbol{\Delta\eta}^r = \boldsymbol{\Delta\eta} - \boldsymbol{\Delta\eta}_0$ is the deviation of the perturbed system from the reference trajectory ($\boldsymbol{\Delta\eta}$) minus the base response ($\boldsymbol{\Delta\eta}_0$) and $< c \,|$ is a standard basis vector selecting a single component of $\boldsymbol{\Delta\eta}^r(t)$.[2] Here $\Delta\theta_0 = < c \,|\; \boldsymbol{\Delta\eta}_0$ is the prescribed perturbation and $M^\flat(t) + |\, b(t) >< c \,| =$

[1]cf http://www.lmd.jussieu.fr/ZOOM/

[2]Dirac bra-ket notation ($< |, | >$) is used for row and column vectors to be consistent with both LAH:I and LAH:II and, as there, because the scalars can clearly be identified in the algebraic developments.

$M(t) = \partial_\eta \boldsymbol{g}(t)$ is the linearized system's Jacobian. This system is in the feedback form which describes the response to the prescribed perturbation. It is linear because the Jacobian matrices are known independently of the unknown vector $\boldsymbol{\Delta\eta}^r$. The general solution of the effective response system is

$$\boldsymbol{\Delta\eta}^r(t) = \Phi(t,0)\boldsymbol{\Delta\eta}^r(0) + \int_0^t \Phi(t,\tau)\,|\,b(\tau)\!> \Delta\theta_0(\tau)d\tau$$

where $\Phi(t,\tau)$ is the propagator (or the state-transition matrix). The state-vector effective response is defined as the state response in (1) to the basic perturbation with no initial perturbation; $\boldsymbol{\Delta\eta}^r(0) = 0$, so

$$\boldsymbol{\Delta\eta}^r(t) = \int_0^t \Phi(t,\tau)\,|\,b(\tau)\!> \Delta\theta_0(\tau)\,d\tau \tag{2}$$

and correspondingly the effective feedback component is defined as

$$\Delta\theta(t) = <\!c\,|\; \boldsymbol{\Delta\eta}^r(t).$$

It is the presence of the propagating matrix $\Phi(t,\tau)$ which invites a furthering of the feedback analysis with Floquet theory. The following introduction to the theory highlights a decomposition of $\Phi(t,\tau)$ which suggests climatic interpretations and provides information on the asymptotic behavior of the effective feedback components.

3 Floquet Theory

Floquet theory applies to systems of differential equations for which the corresponding Jacobian matrix, $M(t)$, is periodic. If the period length is T this means $M(t+T) = M(t), \forall t$. As the following demonstration shows, the theory provides useful properties of $\Phi(t,\tau)$, the system's propagating matrix.

To begin let $X(t)$ be a matrix such that $\Phi(t,\tau) = X(t)X^{-1}(\tau)$. Since $\Phi(t,\tau)$ is the solution matrix of the system

$$\partial_t\Phi(t,\tau) = M(t)\Phi(t,\tau)\ , \quad \Phi(\tau,\tau) = I$$

then

$$\partial_t X(t) = M(t)X(t)$$

and, furthermore,

$$\partial_t X(t+T) = M(t)X(t+T)\ .$$

Hence $X(t+T)$, like $X(t)$, is a fundamental set of solutions for the system and may be expressed as $X(t+T) = X(t)H$ for some constant matrix H. Thus in a certain way the periodicity of $M(t)$ is passed on to the propagator;

$$\begin{aligned}\Phi(t+T, \tau+T) &= X(t+T)X^{-1}(\tau+T) \\ &= X(t)HH^{-1}X^{-1}(\tau) \\ &= \Phi(t,\tau) \ , \forall t \ .\end{aligned}$$

The equality $X(t+T) = X(t)H$ also implies for the propagator that

$$\Phi(t+T, t) = X(t+T)X^{-1}(t) = X(t)HX^{-1}(t) \ , \forall t \ . \tag{3}$$

Equation (3) demonstrates that for all t the transition matrix on one period, $\Phi(t+T,t)$, is similar to H and thus they have the same eigenvalues. Letting $t=0$ and writing the eigendecomposition of H in terms of T; $H = Pe^{\Lambda T}P^{-1}$ with Λ both complex and diagonal and hence $e^{\Lambda T}$ diagonal, this implies further that

$$\Phi(T,0)Z = Ze^{\Lambda T} \tag{4}$$

where $Z = X(0)P$. So $Z = [z_{ik}]$ is both the matrix of right eigenvectors for $\Phi(T,0)$ as well as a fundamental set of solutions at $t=0$. The coefficients of Λ are called the Floquet exponents and it will be assumed their real parts are negative (the system is stable) and arranged in decreasing order. Each entire eigenvalue $e^{\lambda_k T}$ is called a Floquet multiplier. These eigen elements of $\Phi(T,0)$ are essential to Floquet Theory. They will be referred to collectively as Floquet elements.

From Z yet another fundamental solution matrix $Y(t) = [y_{ik}(t)]$ may be defined by

$$\begin{aligned}Y(t) &\stackrel{def}{=} \Phi(t,0)\, Z \ , \\ Y(0) &= Z \ .\end{aligned}$$

Then the relation between $Y(t)$ and its replica at $t+T$ is

$$Y(t+T) = \Phi(t+T,0)\, Z = \Phi(t+T,T)\, \Phi(T,0)\, Z = \Phi(t,0)\, Z\, e^{\Lambda T} = Y(t)\, e^{\Lambda T} \ .$$

So over any period interval each k^{th} column of Y is damped by $e^{\lambda_k T}$, where λ_k is the k^{th} coefficient of Λ. The Floquet vectors, $\Psi^F(t) = \left[\psi^F_{ik}(t)\right]$, are retrieved finally by preventing the damping of Y,

$$\Psi^F(t) \stackrel{def}{=} Y(t)e^{-\Lambda t} = \Phi(t,0)Ze^{-\Lambda t} \ . \tag{5}$$

With this definition, the Floquet vectors are easily seen to be periodic,

$$\begin{aligned}\Psi^F(t+T) &= Y(t+T)e^{-\Lambda T}e^{-\Lambda t} \\ &= Y(t)e^{-\Lambda t} \\ &= \Psi^F(t) \ .\end{aligned}$$

From the definition of the Floquet vectors in equation (5) also comes the equality $\Phi(t,0) = \Psi^F(t)e^{\Lambda t}Z^{-1}$ and thus

$$\begin{aligned}\Phi(t,\tau) &= \Phi(t,0)\Phi^{-1}(\tau,0) \\ &= \Psi^F(t)e^{\Lambda t}Z^{-1}[\Psi^F(\tau)\,e^{\Lambda\tau}Z^{-1}]^{-1} \\ &= \Psi^F(t)e^{\Lambda t}Z^{-1}Ze^{-\Lambda\tau}\Psi^{-F}(\tau) \\ &= \Psi^F(t)e^{\Lambda(t-\tau)}\Psi^{-F}(\tau) \ .\end{aligned}$$

It is this decomposition

$$\Phi(t,\tau) = \Psi^F(t)e^{\Lambda(t-\tau)}\Psi^{-F}(\tau) \ , \tag{6}$$

which can be written also as $\sum_k e^{\lambda_k(t-\tau)} \mid \psi_k^F(t) >< \psi_k^{-F}(\tau) \mid$, that allows for a determination of the asymptotic behavior of the effective feedback components.

3.1 Floquet Theory in Feedback Analysis

As described in LAH:II, Floquet theory furthers the formal feedback analysis by providing both a link to climatic interpretations as well as information on the asymptotic cycle of the effective response.

The climate interpretation comes from viewing the eigenvalues in terms of their characteristic times and their respective Floquet vectors as describing the set of climate mechanisms which correspond to those characteristic times. With this view the vectors of Z describe the initial perturbations to each of these mechanisms. In this way the separate effects of rapid, moderate and slow mechanisms can be deciphered in the feedback analysis and the phase space decomposed into a direct sum of Floquet subspaces associated with the characteristic times.

Considering only periodically forced stable systems, the asymptotic behavior of the effective feedback components can be determined by plugging the decomposition of $\Phi(t,\tau)$ from (6) into equation (2) - a complete derivation can be found in LAH:II;

$$\lim_{n\to\infty} \Delta\theta(t+nT) = \sum_k \frac{1}{1-e^{\lambda_k T}}\Delta\bar{\theta}_k(t)$$

where

$$\Delta\overline{\theta}_k(t) = <c\,|\,\psi_k^F(t)> \int_t^{t+T} e^{\lambda_k(t+T-\tau)} < \psi_k^{-F}(\tau)\,|\,b(\tau)> \Delta\theta_0(\tau) d\tau \ .$$

So mechanisms of all characteristic times are constantly perturbed and adding to the limit feedback cycle. It is this need for even the fast dying mechanisms that proves difficult numerically. In the previous method $\Psi^F(t)$ is calculated directly from the definition given in equation (5). With this method, since the components of $\Phi(t,0)$ which correspond to fast dying eigenvalues are inaccurately determined by the end of the period, it is numerically tractable to determine only those Floquet vectors associated to Floquet multipliers still alive at the end of the first period.
In the case of the simple climate model being used, the standard method accurately determines only the first and second Floquet exponents and vectors. The Floquet exponents are $\lambda_1 = -0.000794823$ per day and $\lambda_2 = -0.009000927$ per day respectively. They correspond to characteristic times of about 3.5 years for the first Floquet multiplier and 3.7 months for the second. The third and fourth Floquet multipliers, which are both less then 10^{-14} at the end of the first period, are damped too quickly for accurate values to be taken from the results of the standard method. A new method is needed.

4 Weighted Projection Method

4.1 Essential Ideas

Amplification along time of the rapidly damped Floquet multipliers provides an intuitive solution for retrieval of the so far undetermined Floquet vectors. The decomposition of $\Phi(t,0)$ into the sum of spaces spanned by each of the Floquet elements,

$$\Phi(t,0) = \sum_k e^{\lambda_k t} |\psi_k^F(t)> < z_k^{-1}| \ , \tag{7}$$

highlights a means for such an amplification. From this view, the amplification may be implemented by creating a new propagating matrix, call it $\Phi^\rho(t,0)$, in which a weighted projection on the true propagator, $\Phi(t,0)$, amplifies continuously the spaces spanned by the rapidly damped solution vectors;

$$\Phi^\rho(t,0) \overset{def}{=} \Phi(t,0) Z e^{Kt} Z^{-1} \ . \tag{8}$$

Here K is a diagonal matrix of the same dimension as the system which simultaneously selects the spaces to be amplified and designates the magnitude of the amplification. From now on the term $Ze^{Kt}Z^{-1}$ will be referred to as the projector. Those spaces spanned by Floquet vectors already accurately determined need not be modified and thus the corresponding diagonal entries of K are set to 0. The rapidly damped spaces, however, are selected for amplification by a non zero entry in the corresponding entries of K. All non-zero entries of K have the same value, let it be denoted by ρ, and thus each amplification has a time dependent magnitude of $e^{\rho t}$. Thus, based on earlier assumptions regarding the eigenvalues ordering, the matrix K in its general form is $K = [0 \ldots 0\ \rho \ldots \rho]_{\text{diag}}$. Expanding $\Phi(t, 0)$ in the definition of $\Phi^{\rho}(t, 0)$ and again simplifying it becomes evident that the new projection matrix is similar to the true projection matrix, however, with selected eigenvalues amplified;

$$\Phi^{\rho}(t, 0) = \Phi(t, 0) Z e^{Kt} Z^{-1} = \Psi^{F} e^{\Lambda t} Z^{-1} Z e^{Kt} Z^{-1} = \Psi^{F} e^{(\Lambda + K)t} Z^{-1}.$$

Thus this new propagator changes the system only by extending the characteristic times of rapid mechanisms. The significance of this is highlighted at time $t = T$,

$$\Phi^{\rho}(T, 0) Z = Z e^{(\Lambda + K)T} \ ,$$

where the fundamental eigen elements of equation (4) remain "visible" and even the same except for the amplified eigenvalues.
It may be noted that this solution may be expressed equivalently as

$$\Phi^{\rho}(t, 0) = \Psi^{F} e^{Kt} \Psi^{-F} \Phi(t, 0) \tag{9}$$

since

$$\begin{aligned}
\Psi^{F} e^{Kt} \Psi^{-F} \Phi(t, 0) &= \Psi^{F} e^{Kt} \Psi^{-F} \Psi^{F} e^{\Lambda t} Z^{-1} \\
&= \Psi^{F} e^{(K + \Lambda)t} Z^{-1} \\
&= \Psi^{F} e^{\Lambda t} Z^{-1} Z e^{Kt} Z^{-1} \\
&= \Phi(t, 0) Z e^{Kt} Z^{-1} \, .
\end{aligned}$$

4.2 The Method

With this new projection matrix the Floquet vectors not numerically tractable with the standard method may be determined successively in order of decreasing characteristic times as follows. At each step the code used to implement the method in Mini_ker for the simple model is included.

Suppose the first k–1 Floquet exponents and vectors are already determined accurately. Then the $k^{\rm th}$ elements must be determined only in terms of this accurate information. With this in mind the projector may be defined as

$$Ze^{Kt}Z^{-1} = \left[I - |z_1 >< z_1^{-1}| - \cdots - |z_{k-1} >< z_{k-1}^{-1}|\right] e^{\rho t} \\ + |z_1 >< z_1^{-1}| + \cdots + |z_{k-1} >< z_{k-1}^{-1}| \,. \quad (10)$$

Mini_ker

In Mini_ker the necessary information is read and the basic elements of the projector are calculated in ZINIT; a portion of the full Mini_ker code that is executed once at the beginning of the run. These basic elements are

$$\mathrm{Projk} = \frac{|\, z_k >< z_k^{-1} \,|}{< z_k^{-1} \,|\, z_k >}$$

where the denominator is included to account for numerical inaccuracies in the identity $ZZ^{-1} = I$. Afterwards the full projector is calculated in ZSTEP; another portion of the Mini_ker code that is executed after each time step.

```
+SELF,IF=FLOQUET2.
;
...
;
!initialise Z^(-1)
OPEN (50, File='floVPl.init', status='OLD');  "is still the transpose"
     READ (50,1009) tata, ((Flo_Zinv(i,j),tata,i=1,np),j=1,np);
      CLOSE(50);
      Z_pr/Z inverse/: ((j,Flo_Zinv(i,j), i=1,np),j=1,np);
;
Do i=1,n
 <zeta_1(i)=Flo_Z(i,1);         "|z_1>"
 >;
Do i=1,n
 <zeta_2(i)=Flo_Z(i,2);         "|z_2>"
 >;
Do i=1,n
 <zeta_3(i)=Flo_Z(i,3);         "|z_3>"
 >;
Do i=1,n
 <zeta_4(i)=Flo_Z(i,4);         "|z_4>"
 >;
;
!========================
! for finding psi_2, psi_3, psi_4
!========================
sprod1=0. ;
Do i=1,n
 <sprod1=sprod1+Flo_Zinv(i,1)*Flo_Z(i,1);
```

```
 >;
Do((i=1,n), j=1,n)
 <Proj1(i,j)=Flo_Z(i,1)*Flo_Zinv(j,1);
 >;
call scamat(1/sprod1,Proj1,Proj1,n,np,n,np);
;
!========================
! for finding psi_3, psi_4
!========================
sprod2=0. ;
Do i=1,n
 <sprod2=sprod2+Flo_Zinv(i,2)*Flo_Z(i,2);
 >;
Do((i=1,n), j=1,n)
 <Proj2(i,j)=Flo_Z(i,2)*Flo_Zinv(j,2);
 >;
call scamat(1/sprod2,Proj2,Proj2,n,np,n,np);
;
!========================
! for finding psi_4
!========================
sprod3=0. ;
Do i=1,n
 <sprod3=sprod3+Flo_Zinv(i,3)*Flo_Z(i,3);
Do((i=1,n), j=1,n)
 <Proj3(i,j)=Flo_Z(i,3)*Flo_Zinv(j,3);
 >;
call scamat(1/sprod3,Proj3,Proj3,n,np,n,np);
;
```

Now in ZSTEP, below is the code used to calculate the projector to determine the third Floquet vector. The projectors used to determine the second and fourth Floquet vectors are built similarly.

```
!++++++++++++++++++
+SELF, IF=FLOQUET2.
!++++++++++++++++++
;
scal = exp(rho*dt);
;
!=====================
!calculate the projector to determine psi_3
!  := Z*exp(Rho*dt)*Z^(-1)
!   = [I-|z_1><z_1^(-1)|/<z_1^(-1)|z_1>-|z_2><z_2^(-1)|/<z_2^(-1)|z_2>]e^(rho*dt)
!     +|z_1><z_1^(-1)|/<z_1^(-1)|z_1>+|z_2><z_2^(-1)|/<z_2^(-1)|z_2>
!------------------
 call initmat(0. ,Proj,n,np,n,np); "clear Proj"
 Do i=1,n
  <Proj(i,i)=1. ;
  >;
 Do((i=1,n), j=1,n)
  <Proj(i,j)=Proj(i,j)-Proj1(i,j)-Proj2(i,j);
  >;
 call scamat(scal,Proj,Proj,n,np,n,np);
 Do((i=1,n), j=1,n)
  <Proj(i,j)=Proj(i,j)+Proj1(i,j)+Proj2(i,j);
  >;
```

;

Once the appropriate projector has been calculated the determination of the Floquet vector is a two step process. The first step is to determine $| z_k >$. This may be done by advancing $\Phi^\rho(t,0) = \Phi(t,0)Ze^{Kt}Z^{-1}$ with the projector in (10) along the first period. Although the definitions (8) and (9) are equivalent the former is used to determine the advance of Φ^ρ. This decision stems from both the ease of computing with the constant eigenvectors of the Z matrix as opposed to the time dependent Floquet Vectors as well as the availability of equations which have been found for advancing the first expression of the propagator. Since $\Phi^\rho(0,0) = I$ is known,

$$\Phi^\rho(0,0) = \Psi^F(0)e^{(\Lambda+K)0}Z^{-1} = ZIZ^{-1} = I \ ,$$

the new propagator Φ^ρ may be advanced iteratively using the final following equation,

$$\begin{aligned}
\Phi^\rho(t+dt,0) &= \Phi(t+dt,0)Ze^{K(t+dt)}Z^{-1} \\
&= \Phi(t+dt,t)\Phi(t,0)Ze^{Kt}Z^{-1}Ze^{Kdt}Z^{-1} \\
&= e^{M(t)dt}\Phi^\rho(t,0)Ze^{Kdt}Z^{-1} \\
&\qquad\qquad \text{if } dt \text{ is taken small enough} \\
&= \left[\frac{I+M(t)\frac{dt}{2}}{I-M(t)\frac{dt}{2}}\right]\Phi^\rho(t,0)Ze^{Kdt}Z^{-1} \\
&\qquad\qquad \text{using a Padé approximant of the exponential.}
\end{aligned}$$

Once Φ^ρ has been advanced along the entire period since

$$\Phi^\rho(T,0) = Ze^{(\Lambda+K)T}Z^{-1}$$

then $| z_k >$ may be found through an eigendecomposition of $\Phi^\rho(T,0)$. Note if $k < n$, where n is the dimension of K, then $< z_k^{-1} |$ should also be determined from this decomposition in order that it may be used in the next projector(s).

Mini_ker

The modified projector is advanced in ZSTEP;

```
if (time.eq.0.)
< call INITMAT(0.,Phi_Rho,n,np,n,np);
  Do i=1,n
  <Phi_Rho(i,i)=1.;
   >;
```

```
>;
! =========================================
! advance of Phi_Rho by aspha where aspha:=M(t)
! Phi_Rho(t+dt)=exp(aspha*dt)Phi_Rho*Proj
!  exp(aspha*dt)=(I+(dt/2)aspha))/(I-(dt/2)aspha)
! =========================================
! Calculation of I-dt/2aspha
! ----------------------
call oscamat(-dt/2.,n,n,aspha,np,padDen,np);
Do i=1,n
< padDen(i,i)=1.+padDen(i,i);
>;
;
! Product (I+dt/2aspha)Phi_Rho(Proj)
! -------------------------
call oscamat(dt/2.,n,n,aspha,np,padNum,np);
Do i=1,n
< padNum(i,i)=1.+padNum(i,i);
>;
call matmlt(Phi_Rho,Proj,Phi_Rho,n,np,n,np,n,np);
call omatmlt(n,n,n,padNum,np,Phi_Rho,np,Phi_Rho,np);
call Sgesv(np,np,padDen,np,iPiv,Phi_Rho,np,infores);
;
```

Now, using the well determined $|\, z_k >$

$$|\, \psi_k^F(t) >= \Phi^\rho(t,0)\,|\, z_k > e^{-(\lambda_k+\rho)t}$$

may be determined along the entire first period. Choosing $\rho = -\lambda_k$ simplifies the results further;

$$|\, \psi_k^F > (t) = \Phi^\rho(t,0)\,|\, z_k > \ .$$

With $\rho = -\lambda_k$the space spanned by the $k+1$ Floquet elements is kept from damping at all and the necessary information is retrieved. Finally, the newly determined values of $|\, z_k >$ and $< z_k^{-1}\,|$ may be updated and the procedure repeated for the $k+1^{\text{st}}$ Floquet vector if necessary.

Mini_ker

Portions of the following code in ZSTEP must be excluded as necessary to coordinate with the value of k.

```
! Calculating Psi^(F)
!============================
!------------
! Calculate |psi_1(t)>=Phi_Rho(t)*|z_1>exp(-lambda(1)t)
!------------
! zeta_1 := |z_1> already calculated in $ZINIT
amp = exp(-Flo_lambda(1)*(time+dt));
call MATVEC(Phi_Rho,zeta_1,y_1,n,np,n,np);
call SCAVEC(amp,y_1,psi_1,np);
;
```

```
! !------------
! ! Calculate |psi_2(t)>=Phi_Rho(t)*|z_2>
! !        IF rho.eq.-lambda(2)
! !------------
! ! zeta_2 := |z_2> already calculated in $ZINIT
! call MATVEC(Phi_Rho,zeta_2,psi_2,n,np,n,np); "calculate psi_2"
! ;
;
!------------
! Calculate |psi_2(t)>=Phi_Rho(t)*|z_1>exp(-lambda(2)t)
!      IF rho.neq.-lambda(2)
!------------
!zeta_2 := |z_2> already calculated in $ZINIT
amp = exp(-Flo_lambda(2)*(time+dt));
call MATVEC(Phi_Rho,zeta_2,y_2,n,np,n,np);
call SCAVEC(amp,y_2,psi_2,np);
;
!------------
! Calculate |psi_3(t)>=Phi_Rho(t)*|z_3>  IF rho=0.080
!------------
! zeta_3 := |z_3> already calculated in $ZINIT
amp = exp(0.008353574*(time+dt));
call MATVEC(Phi_Rho,zeta_3,y_3,n,np,n,np);
call SCAVEC(amp,y_3,psi_3,np);
;
```

4.3 Results From the Simple Climate Model

The weighted projection method was first tested in Mini_ker by using the first Floquet elements and $\rho = -\lambda_2$ to determine the second Floquet vector. Figure 1 below shows the eigenvalues of both $\Phi(t,0)$ and $\Phi^\rho(t,0)$ over the first period. The first eigenvalue (dark blue) remained the same, the third (black) and fourth (light blue) eigenvalues were slightly amplified compared with those of $\Phi(t,0)$, and the second (orange)became periodic with a value of 1 at the end of the period as expected. The results were checked against

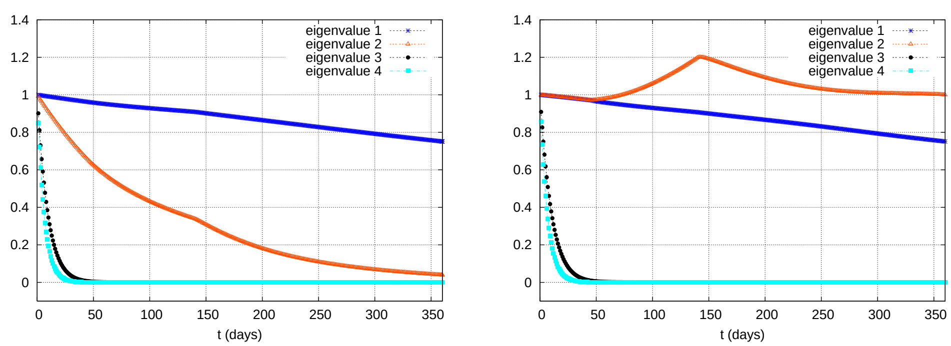

Figure 1: Eigenvalues of $\Phi(t,0)$ (left) and $\Phi^\rho(t,0)$ (right) with $\rho = -\lambda_2$.

those already accurately determined by the standard method and the two methods produced results identical till 5 decimal places. The four components of the first and second Floquet vectors along the first period are shown in figure 2.

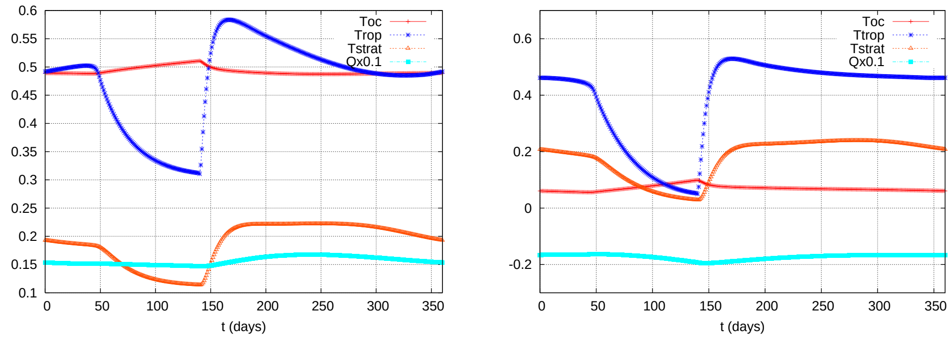

Figure 2: Four components of the first (left) and second (right) Floquet vectors over one period.

Before the new method could be used to determine the third Floquet vector it was necessary to accurately determine the value of $-\lambda_3$. This was done through several runs of Mini_ker in which various values of ρ, beginning with the approximation taken from the standard method, were used to narrow in on the correct amplification. It was determined that $\lambda_3 = -0.088353574$. Thus the characteristic time of the corresponding third Floquet multiplier is just over 11 days. The exact value of λ_3 could not be used, however, as the value of ρ in Mini_ker since the amplification of the third eigenvector also amplified still undetermined model features along the run and the information was skewed by the end of the period. Therefore, rather, ρ was set equal to 0.08 and the Mini_ker code to evaluate the third Floquet vector was adjusted accordingly with an amplification of $e^{0.008353574t}$ along the period;

```
!------------
! Calculate |psi_3(t)>=Phi_Rho(t)*|z_3>  IF rho=0.080
!------------
! zeta_3 := |z_3> already calculated in $ZINIT
amp = exp(0.008353574*(time+dt));
call MATVEC(Phi_Rho,zeta_3,y_3,n,np,n,np);
call SCAVEC(amp,y_3,psi_3,np);
;
```

The left image in figure 3 below shows the four eigenvalues along the period. The color coded legend, the same as in figure 1 above, corresponds to the portion of the graph after 200 days. Before then the colors switch between eigenvalues as the amplified third eigenvalue crosses others throughout the period. The right image shows the four components of the newly found third Floquet vector along the first period. It should be mentioned that the vectors $| z_1 >, < z_1^{-1} |, | z_2 >$ and $< z_2^{-1} |$ determined from the eigendecomposition of $\Phi^\rho(T, 0)$ were scalar multiples of the values determined for those vectors

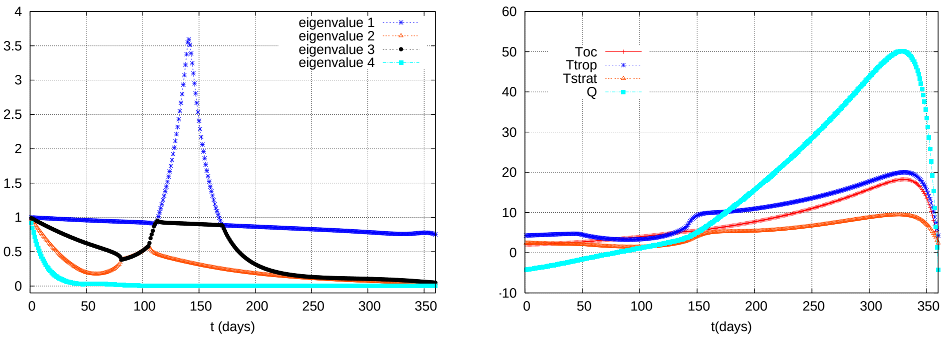

Figure 3: The four eigenvalues (left) with $\rho = 0.08$ and four components of the third Floquet vector (right) along the first period. The exact value of λ_3 is -0.088353574 with a corresponding characteristic time of just over 11 days.

from $\Phi(T,0)$ and $\Phi^{\rho}(T,0)$ when $\rho = \lambda_2$. Thus care must be taken in such a case to use the set of cross normalized $|\, z_k >$ and $< z_k^{-1} \,|$ to find the next Floquet vector instead of only updating the values of $|\, z_3 >$ and $< z_3^{-1} \,|$.
As in the case of the third Floquet vector the amplification necessary to keep $|\, y_4(t) >$ from dying out caused amplification of undetermined model features. In this case, however, due to the extremely rapid damping of $e^{\lambda_4 t}$, the amplification was too great and the information too skewed to utilise the new method in determining the fourth Floquet vector. Since, however, the first three Floquet vectors and exponents are known, calculating the residual of $\Phi^{\rho}(t,0)$ as in (7) with the necessary amplifications applied, which is possible, may provide a means of determining the fourth Floquet vector if explored further.

5 Left Projector

Alongside the work done to determine the evanescent Floquet vectors was an attempt to advance the left projector, $|\, \Psi_k^F(t) >< \Psi_k^{-F}(t) \,|$ along one period. It is called such here since, like each k^{th} projector in (10), it projects onto the space spanned by the k^{th} Floquet elements but this time from the left of the matrix. As described in LAH:II, due to the decomposition of Φ into its Floquet elements as in (6), once all the Floquet vectors are known the left projector must be determined along one period for each k in order to obtain the cyclic response $\Delta\overline{\theta}_k(t)$. Assuming the Floquet vectors have been found, the left projector is known at $t = 0$; $|\, \Psi_k^F(0) >< \Psi_k^{-F}(0) \,| = |\, z_k >< z_k^{-1} \,|$, and thus an attempt was made to advance it step by step along the first period using the following final equality,

$$
\begin{aligned}
|\,\Psi_k^F(t+dt)\!> \,<\!\Psi_k^{-F}(t+dt)\,| \\
&= \Phi(t+dt,o)\,|\,z_k\!> e^{-\lambda_k(t+dt)} e^{\lambda_k(t+dt)} <\!z_k^{-1}\,|\;\Phi^{-1}(t+dt,0) \\
&= \Phi(t+dt,t)\Phi(t,o)\,|\,z_k\!><\!z_k^{-1}\,|\;\Phi^{-1}(t,0)\Phi^{-1}(t+dt,t) \\
&= e^{M(t)dt}\Phi(t,o)\,|\,z_k\!><\!z_k^{-1}\,|\;\Phi^{-1}(t,0)e^{-M(t)dt} \\
&\qquad\qquad \text{if } dt \text{ is small enough} \\
&= e^{M(t)dt}\,|\,\Psi_k^F(t)\!><\!\Psi_k^{-F}(t)\,|\;e^{-M(t)dt} \\
&= \left[\frac{I+M(t)\frac{dt}{2}}{I-M(t)\frac{dt}{2}}\right]|\,\Psi_k^F(t)\!><\!\Psi_k^{-F}(t)\,|\left[\frac{I-M(t)\frac{dt}{2}}{I+M(t)\frac{dt}{2}}\right] \\
&\qquad\qquad \text{using a Padé approximant of the exponential.}
\end{aligned}
$$

When the projector was advanced in Mini_ker, however, it exploded numerically before the end of the period. Looking at another definition of the left projector;

$$|\,\Psi_k^F(t)\!><\!\Psi_k^{-F}(t)\,| = \Phi(t,0)\,|\,z_k\!><\!z_k^{-1}\,|\;\Phi^{-1}(t,0)\ ,$$

illuminates the influence of $\Phi^{-1}(t,0)$ on the projector and thus the cause of the rapid growth. The eigenvalues of $\Phi^{-1}(t,0)$ are the inverses of the eigenvalues of $\Phi(t,0)$ and thus are growing exponentially along the period with very large positive growth constants in the case of the inverses of the third and fourth Floquet exponents. While no successful method was determined for advancing the left projector, those attempts made highlighted that future work to advance the projector must focus on how each Floquet space is behaving and effecting the projector.

6 Conclusion

The new method developed here for determining Floquet vectors associated to rapid mechanisms in order to aid the seasonally influenced climate feedback analysis brought new results in providing a general method for determining Floquet elements of semi-rapid mechanisms and by finding the before indeterminable third Floquet vector, third Floquet exponent and the third eigenvector of Z in the simple climate model. Attempts to determine the fourth Floquet elements as well as the left projector were unsuccessful, however, they provide direction for further work. In the case of the Floquet elements, the amplification of the yet undetermined model features by the weighted projector, overcome in the case of the third Floquet vector, but insurmountable in the case of the fourth Floquet vector, are important to examine in order to advance the weighted projection method. Further work may be done in order to determine whether what was being amplified were numerical inaccuracies or properties of the model itself, and in either case what could be done to avoid their amplification. More work may also be done to determine if the final Floquet vector can be extracted from the residual of the direct sum of Φ^{ρ}. In the case of the left projector further work, as stated earlier, for determining their advance should involve a method attentive to the behavior of the different Floquet spaces. This is due to the undamped exponential growth of $\Phi^{-1}(t,0)$'s eigenvalues in the left projectors. Eventual success in deriving methods for both the numerical determination of all Floquet vectors as well at the left projector would bring useful insight not only to the formal analysis of the effect of the seasonal cycle on climate feedbacks but in other oscillating systems where Floquet theory is used as well.

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