Dynamic Meteorology, Part I

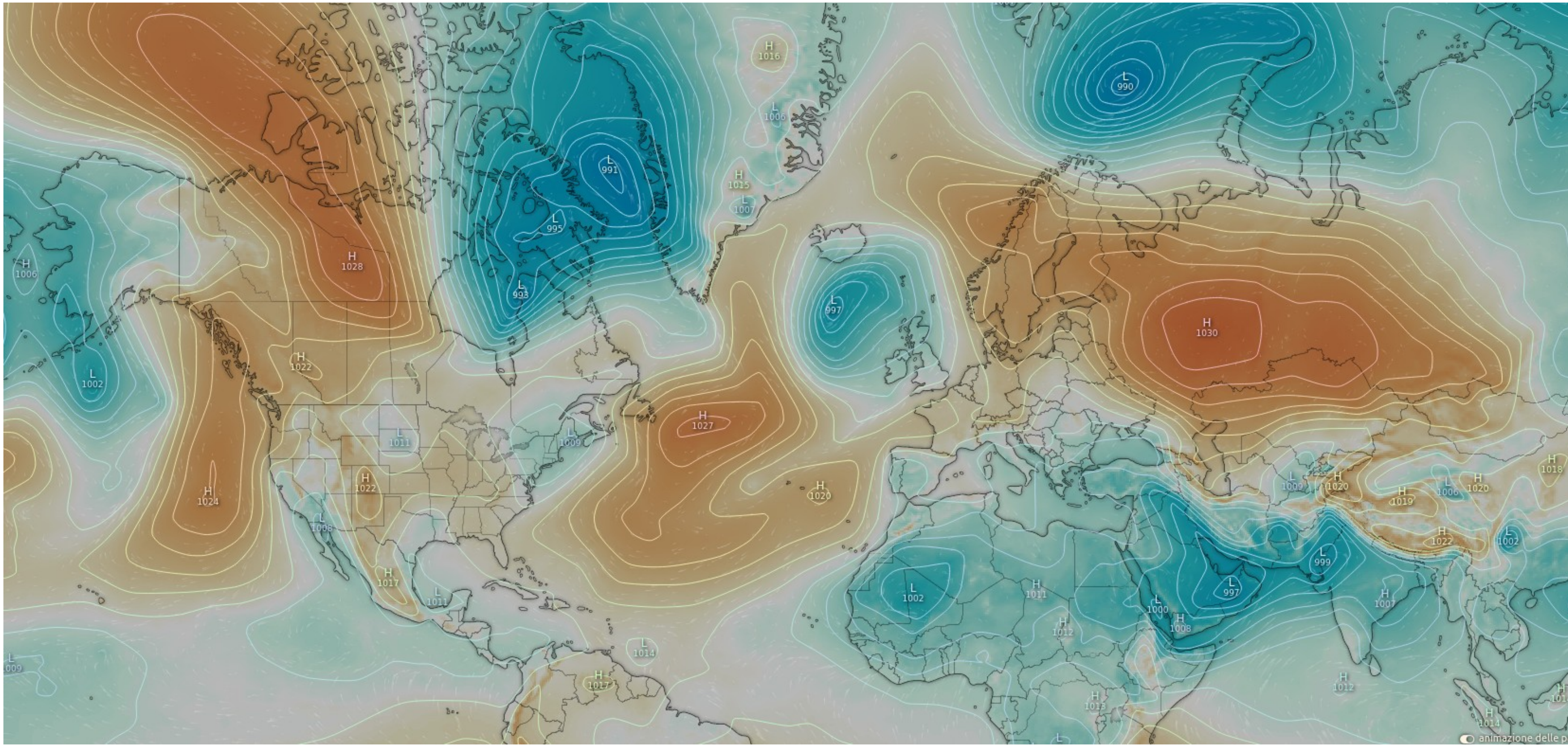
Étienne Vignon

etienne.vignon@lmd.ipsl.fr
Bureau 303, Tour 45-55 3ème étage
Sorbonne Univ., Jussieu

References
« Atmospheric Science » Wallace & Hobbes
« Fondamentaux de Météorologie », Mallardel
« An introduction to dynamic meteorology » de Holton

video jet stream

B.1 Rossby waves



'Planetary scale' waves
→ Rossby waves

Plumb, chap 6

Important remark

In this lecture, the letter ω can refer to either the vertical velocity in pressure coordinate (dp/dt, in the conservation equations) or the pulsation of waves (in the dispersion relationship)

Rossby waves are first of all fluctuations of the 'vorticity'

Relative vorticity wrt Earth rotation
= 2D rotational of the wind

$$\zeta = \partial_x v - \partial_y u.$$

$$\begin{aligned} \frac{du}{dt} - fv &= -\frac{\partial \phi}{\partial x} \\ \frac{dv}{dt} + fu &= -\frac{\partial \phi}{\partial y} \end{aligned}$$

Rossby waves are first of all fluctuations of the 'vorticity'

Relative vorticity wrt Earth rotation :

$$\zeta = \partial_x v - \partial_y u.$$

$$\begin{aligned} \frac{du}{dt} - fv &= -\frac{\partial \phi}{\partial x} \\ \frac{dv}{dt} + fu &= -\frac{\partial \phi}{\partial y} \end{aligned}$$

$$\Downarrow$$

$$\partial_t(\zeta + f) + u\partial_x(\zeta + f) + v\partial_y(\zeta + f) + \omega\partial_p(\zeta + f) + (\zeta + f)(\partial_x u + \partial_y v) + \partial_x\omega\partial_p v - \partial_y\omega\partial_p u = 0$$

In pressure coordinates, the mass conservation equation reads :

$$\partial_x u + \partial_y v + \partial_p \omega = 0$$

Rossby waves are first of all fluctuations of the 'vorticity'

Relative vorticity wrt Earth rotation :

$$\zeta = \partial_x v - \partial_y u.$$

$$\begin{aligned} \frac{du}{dt} - fv &= -\frac{\partial \phi}{\partial x} \\ \frac{dv}{dt} + fu &= -\frac{\partial \phi}{\partial y} \end{aligned}$$

⬇

$$\partial_t(\zeta + f) + u\partial_x(\zeta + f) + v\partial_y(\zeta + f) + \omega\partial_p(\zeta + f) + (\zeta + f)(\partial_x u + \partial_y v) + \partial_x \omega \partial_p v - \partial_y \omega \partial_p u = 0$$

In pressure coordinates, the mass conservation equation reads :

$$\partial_x u + \partial_y v + \partial_p \omega = 0$$

⟶ $$d_t(\zeta + f) - (\zeta + f)(\partial_p \omega) + \partial_x \omega \partial_p v - \partial_y \omega \partial_p u = 0$$

Rossby waves are first of all fluctuations of the 'vorticity'

→
$$d_t(\zeta + f) - (\zeta + f)(\partial_p \omega) + \partial_x \omega \partial_p v - \partial_y \omega \partial_p u = 0$$

We know assume flows at **low Rossby number** (close to geostrophic equilibrium) and a **barotropic atmosphere** (no horizontal gradient of temperature/density)
→ u and v does not vary with pressure (Taylor Proudman theorem) :

$$\frac{d_h}{dt}(\zeta + f) - (\zeta + f)(\partial_p \omega) = 0$$

Rossby waves are first of all fluctuations of the 'vorticity'

→ $$d_t(\zeta + f) - (\zeta + f)(\partial_p \omega) + \partial_x \omega \partial_p v - \partial_y \omega \partial_p u = 0$$

We know assume flows at **low Rossby number** (close to geostrophic equilibrium) and a **barotropic atmosphere** (no horizontal gradient of temperature/density)
→ u and v does not vary with pressure (Taylor Proudman theorem) :

$$\frac{d_h}{dt}(\zeta + f) - (\zeta + f)(\partial_p \omega) = 0$$

If we consider a simplified purely horizontal system → $\partial_p \omega \sim 0$
(2D fluid)

Conservation of absolute vorticity in a barotropic atmosphere :

$$\frac{d_h}{dt}(\zeta + f) = 0$$

Rossby waves are first of all fluctuations of the 'vorticity'

$$d_t(\zeta + f) - (\zeta + f)(\partial_p \omega) + \partial_x \omega \partial_p v - \partial_y \omega \partial_p u = 0$$

We know assume flows at **low Rossby number** (close to geostrophic equilibrium) and a **barotropic atmosphere** (no horizontal gradient of temperature/density)
→ u and v does not vary with pressure (Taylor Proudman theorem) :

$$\frac{d_h}{dt}(\zeta + f) - (\zeta + f)(\partial_p \omega) = 0$$

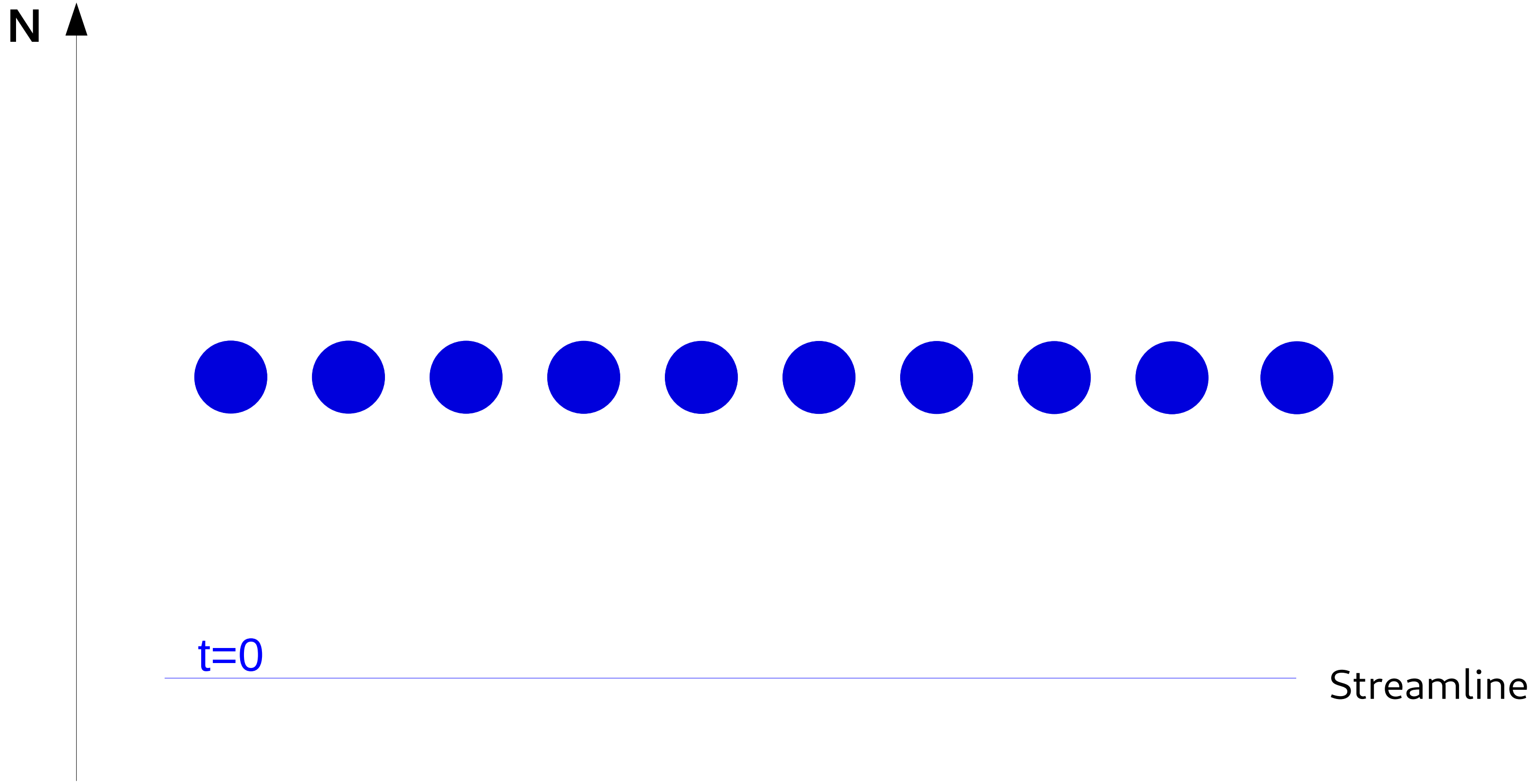
If we consider a simplified purely horizontal system →$\partial_p \omega \sim 0$
(2D fluid)

Conservation of absolute vorticity in a barotropic atmosphere :

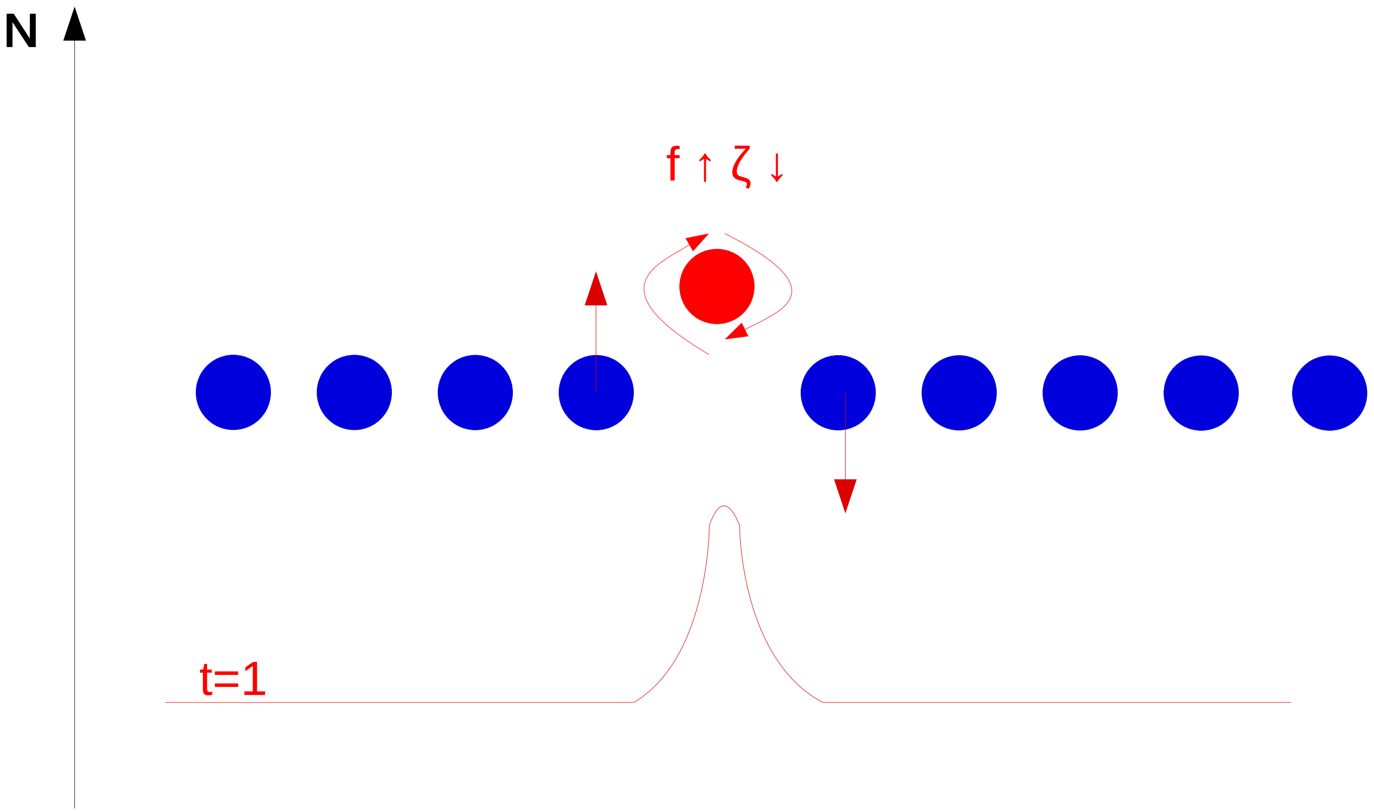
$$\boxed{\frac{d_h}{dt}(\zeta + f) = 0}$$

Relative vorticity planetary vorticity

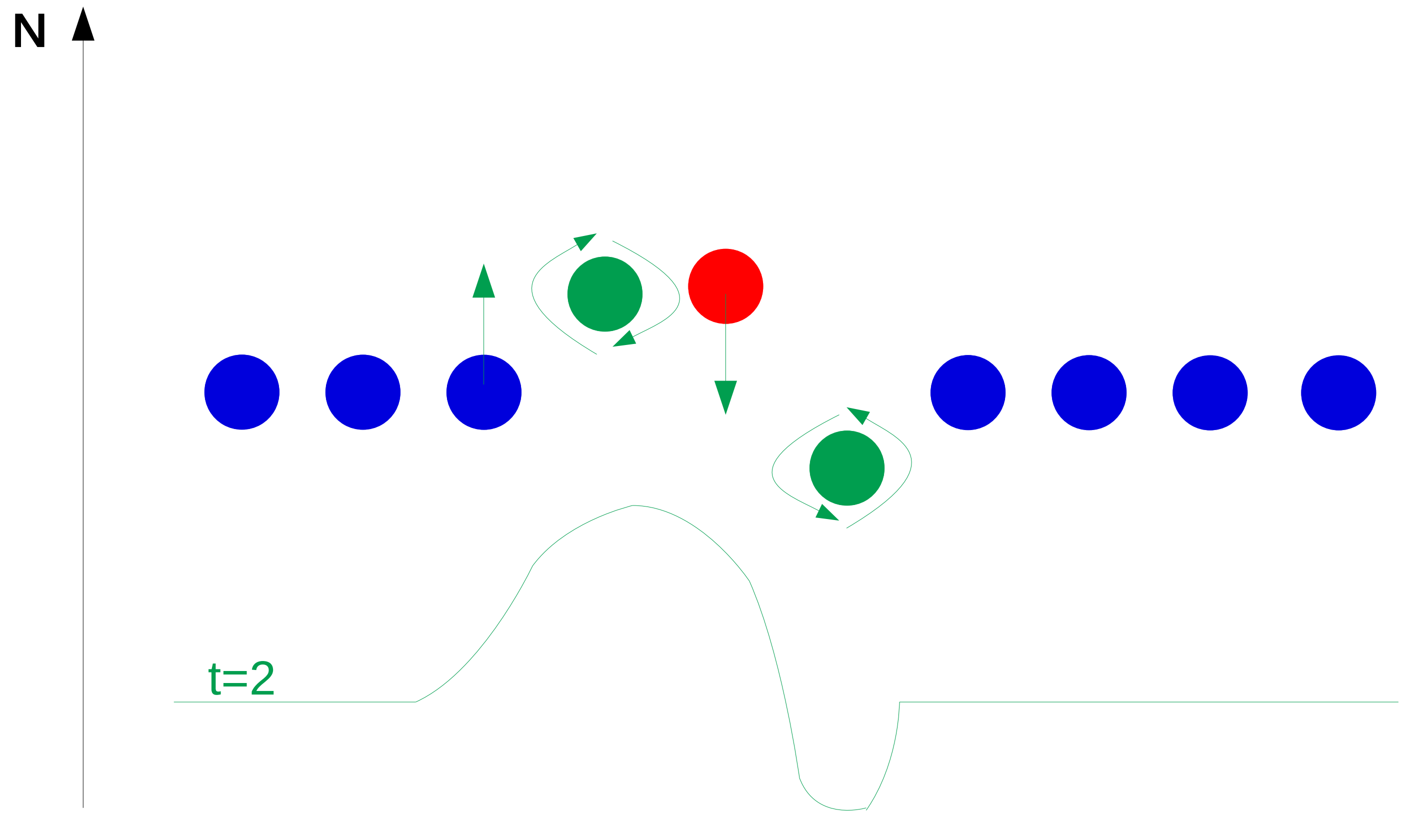
The dance of Rossby waves



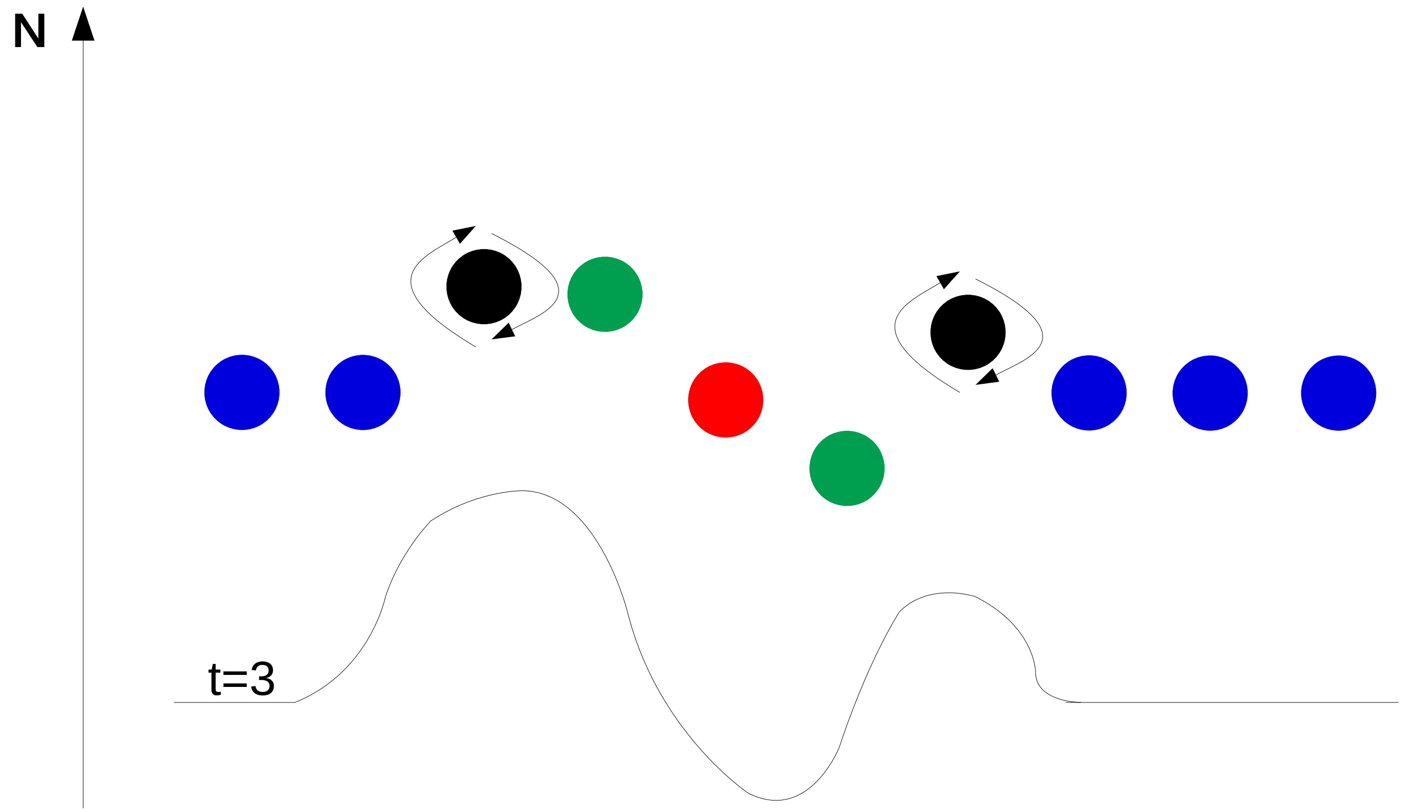
The dance of Rossby waves



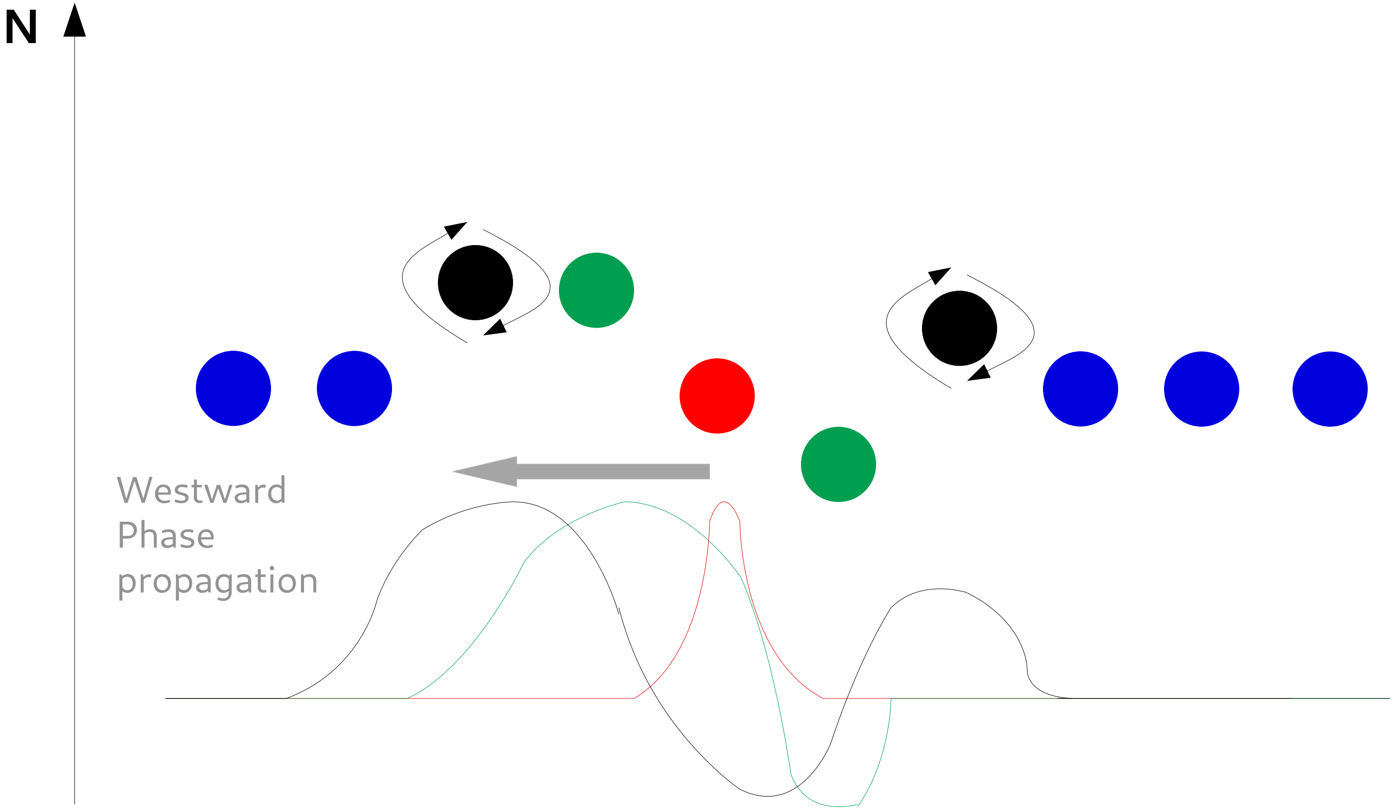
The dance of Rossby waves



The dance of Rossby waves



The dance of Rossby waves



Calculation of the Rossby waves properties
β- plane, 2D barotropic case, Ro<<1

Application of the **so-called perturbation method.**

Separation mean-flow (uniform zonal flow U) and perturbations

$$u = U + u', v = v', \zeta = \partial_x v - \partial y u = \zeta' = \partial_x v' - \partial_y u',$$

Calculation of the Rossby waves properties
β- plane, 2D barotropic case, Ro<<1

Application of the **so-called perturbation method.**

Separation mean-flow (uniform zonal flow U) and perturbations

$$u = U + u', v = v', \zeta = \partial_x v - \partial y u = \zeta' = \partial_x v' - \partial_y u',$$

Conservation absolute vorticity $\frac{d_h}{dt}(\zeta + f) = 0$

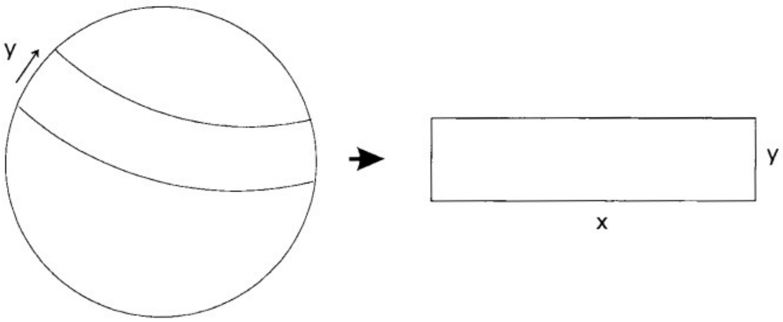
$$\partial_t \zeta' + (U + u')\partial_x \zeta' + v'\partial_y \zeta' + v'\partial_y f = 0$$

Linearization $\partial_t \zeta' + U\partial_x \zeta' + v'\partial_y f = 0$

Calculation of the Rossby waves properties

Rossby waves are synoptic-scale waves (low Rossby numbers)
→ Coriolis parameter f cannot be assumed constant

β-plane framework :



$$f(\phi) \simeq f(\phi_0) + (\phi - \phi_0)\left(\frac{df}{d\phi}\right)(\phi_0) + \ldots$$

$$\begin{aligned} f(\phi_0) &= 2\Omega \sin\phi_0 \; ; \\ df.d\phi_0 &= 2\Omega \cos\phi_0 \; . \end{aligned}$$

$$f(y) = f_0 + \beta y \; ,$$

where $f_0 = 2\Omega \sin\phi_0$ and

$$\beta = \frac{2\Omega}{a}\cos\phi_0 \; .$$

β=df/dy

Calculation of the Rossby waves properties
β- plane, 2D barotropic case, Ro<<1

$$\partial_t \zeta' + U\partial_x \zeta' + v'\underbrace{\partial_y f}_{=cste=\beta} = 0$$

=cste=β

Definition of a stream function

$$\nabla_h \vec{u} = 0. \longrightarrow \quad u = -\partial_y \psi \qquad v = \partial_x \psi \qquad \zeta = \Delta\psi$$

Equation on stream function :

$$\partial_t\left(\frac{\partial^2\psi'}{\partial x^2} + \frac{\partial^2\psi'}{\partial y^2}\right) + U\partial_x\left(\frac{\partial^2\psi'}{\partial x^2} + \frac{\partial^2\psi'}{\partial y^2}\right) + \frac{\partial\psi'}{\partial x}\beta = 0$$

Calculation of the Rossby waves properties
β- plane, 2D barotropic case, Ro<<1

Equation on stream function :

$$\partial_t\left(\frac{\partial^2\psi'}{\partial x^2}+\frac{\partial^2\psi'}{\partial y^2}\right)+U\partial_x\left(\frac{\partial^2\psi'}{\partial x^2}+\frac{\partial^2\psi'}{\partial y^2}\right)+\frac{\partial\psi'}{\partial x}\beta=0$$

Equation linear in x, y and t. Look for a spectral monochromatic wave solution (total solution can be reconstructed from them by Fourier transform, see chap 7.2.1 in Holton) :

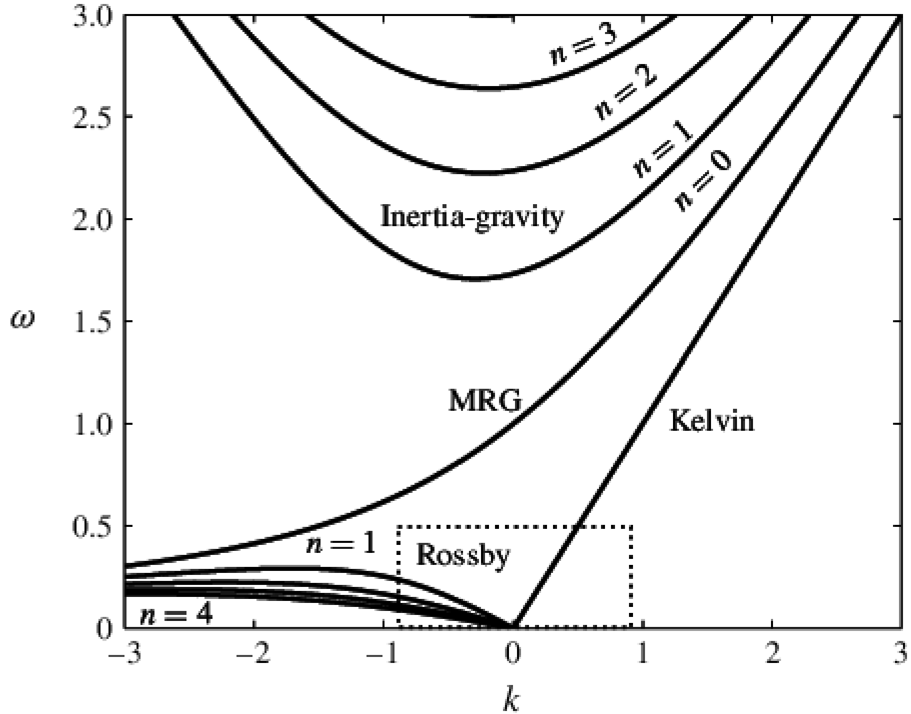
$$\psi'=\Psi e^{i(\omega t-kx-ly)}$$

$$\Rightarrow \quad -i\omega(k^2+l^2)+ikU(k^2+l^2)-ik\beta=0$$

dispersion relationship of Rossby waves

$$\boxed{\omega=Uk-\frac{k\beta}{k^2+l^2}}$$

Calculation of the Rossby waves properties
β- plane, 2D barotropic case, Ro<<1



dispersion relationship of Rossby waves

$$\omega = Uk - \frac{k\beta}{k^2 + l^2}$$

Calculation of the Rossby waves properties
β- plane, 2D barotropic case, Ro<<1

Zonal phase speed

$$c_\phi = \omega / k : \qquad c_\phi = U - \frac{\beta}{k^2 + l^2}$$

➡ **westward** phase propagation wrt mean flow
Phase speed depend on wave number → **dispersion**
Longer waves (smaller k^2) travel faster

Calculation of the Rossby waves properties
β- plane, 2D barotropic case, Ro<<1

Typical values :

At 45 degN,

$$\beta = \frac{2\Omega}{a}\cos 45\,\text{deg} = \frac{2\pi\sqrt{2}}{86400 \times 6.37 \times 10^6} = 1.6145 \times 10^{-11}\text{m}^{-1}\text{s}^{-1} .$$

A typical midlatitude disturbance might have a half-wavelength of 5000km in both directions, so

$$k = l \simeq \frac{\pi}{5 \times 10^6} = 6.28 \times 10^{-7}\text{m}^{-1}$$

and then

$$\omega = -\frac{1.6145 \times 10^{-11}}{2 \times 6.28 \times 10^{-7}} = -1.29 \times 10^{-5}\text{s}^{-1} .$$

The period is

$$\frac{2\pi}{|\omega|} = \frac{2\pi}{1.29 \times 10^{-5}} = 4.87 \times 10^5\text{s}$$

$$\simeq \frac{4.87 \times 10^5}{86400} = 5.6\text{d} .$$

The westward phase speed is

$$c = -\frac{\omega}{k} = \frac{1.29 \times 10^{-5}}{6.28 \times 10^{-7}} = 20.5\text{ms}^{-1} .$$

Calculation of the Rossby waves properties
β- plane, 2D barotropic case, Ro<<1

Zonal phase speed

$$c_\phi = \omega / k : \qquad c_\phi = U - \frac{\beta}{k^2 + l^2}$$

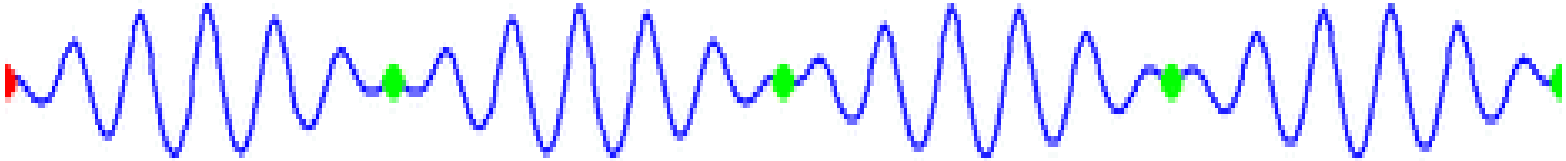
➡ **westward** phase propagation wrt mean flow
Phase speed depend on wave number → **dispersion**
Longer waves (smaller k^2) travel faster

Stationary modes $(c_\phi = 0)$: $\kappa = \sqrt{\beta / U}$

with $\kappa = \sqrt{k^2 + l^2}$ horizontal wave number.

$$2\pi / \kappa \approx 9000 \text{ km}$$

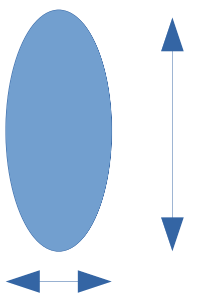
NB : phase and group speeds of a wave



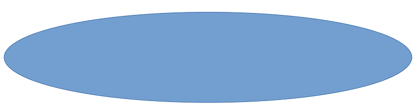
Calculation of the Rossby waves properties
β- plane, 2D barotropic case, Ro<<1

Zonal group speed

$$c_g = \partial\omega/\partial k : \quad c_g = U + \frac{\beta(k^2 - l^2)}{(k^2 + l^2)^2}$$



Lx << Ly, k>>l, the energy propagates eastward wrt mean flow

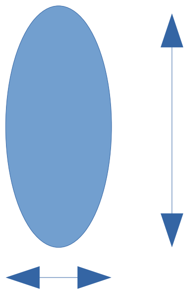


Lx >> Ly, k<<l, the energy propagates westward wrt mean flow

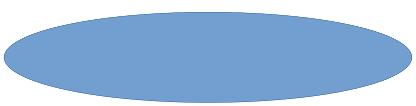
Calculation of the Rossby waves properties
β- plane, 2D barotropic case, Ro<<1

Zonal group speed

$$c_g = \partial\omega/\partial k : \quad c_g = U + \frac{\beta(k^2 - l^2)}{(k^2 + l^2)^2}$$



Lx << Ly, k>>l, the energy propagates eastward wrt mean flow



Lx >> Ly, k<<l, the energy propagates westward wrt mean flow

Group speed of stationary modes
Eastward zonal groupe velocity
Planetary perturbations far from sources

$$c_g(c_\phi = 0) = 2k^2 \frac{U}{\beta}$$

~ 2 m/s

Hovmöller diagrams

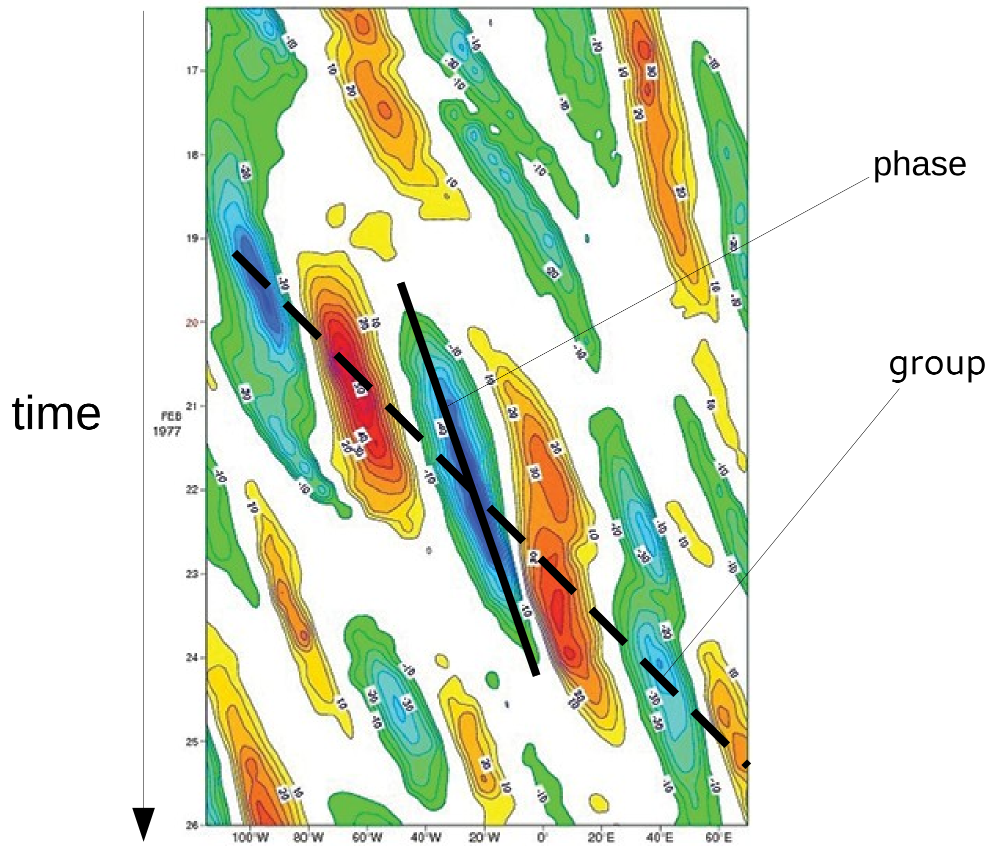


FIG. 6. A modern version of the Hovmöller diagram for the same period as in Fig. 5 (Feb 1977). Instead of 500-hPa geopotential values, the average 250-hPa meridional wind vector is used. The latitude interval is 30°–50°N. A second, weaker downstream development is seen starting on 22 Feb, a couple of days after the first starting on 19 Feb.

Persson 2017

NB : Practical session to estimate Rossby wave properties from Hovmuller diagrams

Relief effects and stationary modes

$$\frac{d_h}{dt}(\zeta + f) - (\zeta + f)(\partial_p \omega) = 0$$

Let's assume a constant density fluid (ρ=constant)

dp=-ρgdz
ω=-ρgw

After vertical integration :

$$(z_2 - z_1)\frac{d_h}{dt}(\zeta + f) = (\zeta + f)(w(z_2) - w(z_1))$$

$$\text{with } H = z_2 - z_1$$

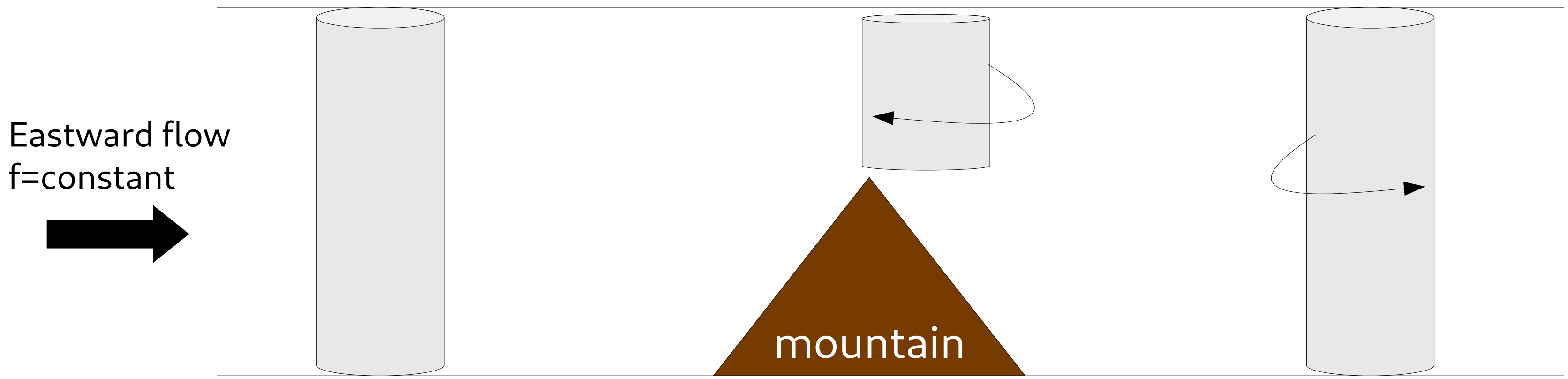
$$\frac{1}{(\zeta + f)}\frac{d_h}{dt}(\zeta + f) = \frac{1}{H}\frac{d_h}{dt}H$$

$$\frac{d_h}{dt}\frac{\zeta + f}{H} = 0$$

Conservation of 'potential vorticity'

Relief effects and stationary modes

$$\frac{d_h}{dt}\frac{\zeta + f}{H} = 0$$



→ 'vortex stretching'
→ Through downstream of a relief
→ More generally, flow divergence (vertical velocity can be viewed as a source of Rossby waves

Relief effects and stationary modes

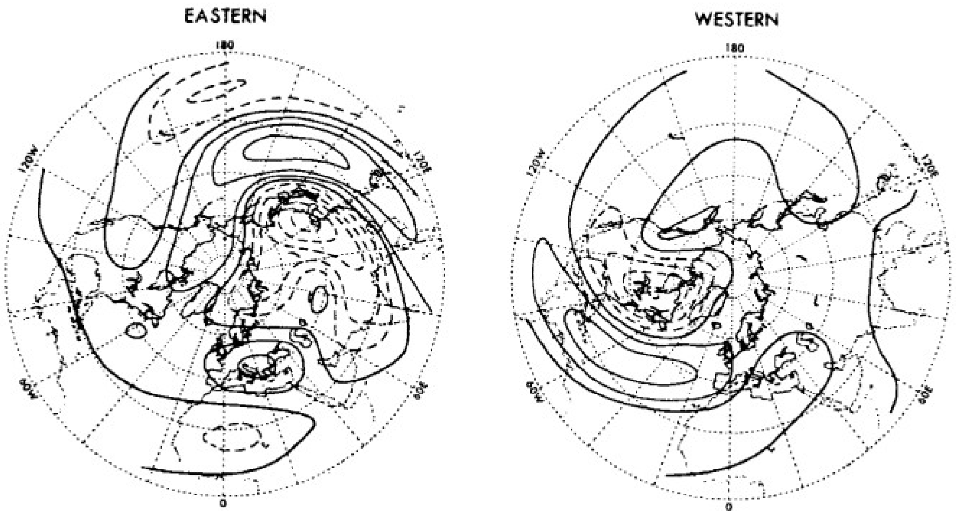
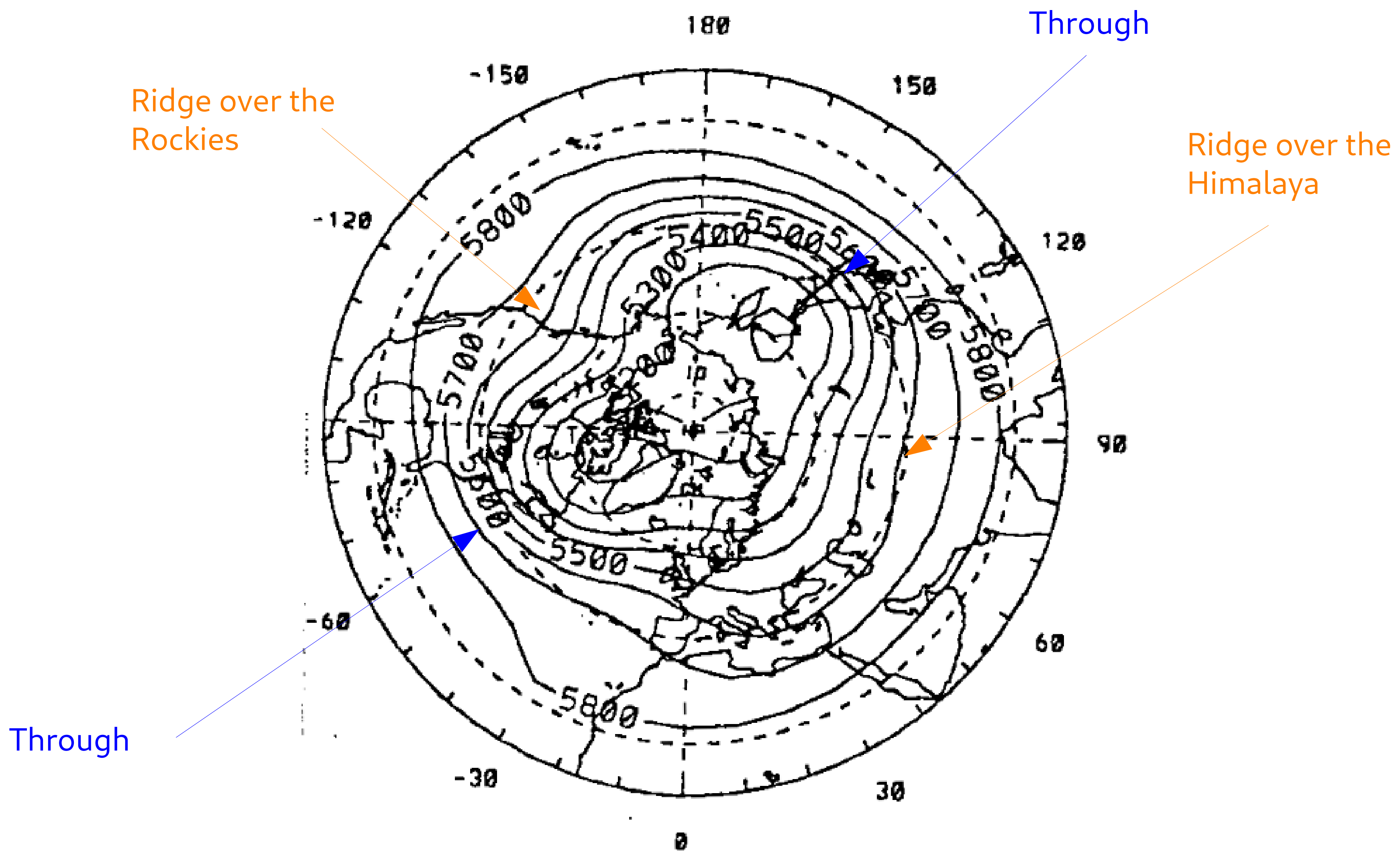


Figure 6.7: Flow over a localized mountain. Numerical solutions for the perturbation streamfunction ψ' for flow over (left) mountains in the eastern hemisphere (Tibet, mostly, with a small contribution from the Alps) and (right) the western hemisphere (mostly the Rockies). ote the Rossby waves propagating "downstream" (eastward) of the mountains.

Relief effects and stationary modes DETAILS in PART II



Stationary modes in the northern hemisphere dominated by the k=2 mode
(two major mountain ranges)

Plumb, chap 6

NB : Ertel's potential vorticity

It can be shown in the (almost) general atmospheric case (baroclinic), the following quantity is conserved :

$$\mathrm{PV} = \frac{(\zeta + f).\nabla\theta}{\rho}$$

for frictionless motions and absence of diabatic processes