

A buoyancy-vorticity wave interaction approach to stratified shear flow. *

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Abstract

Motivated by the success of potential vorticity (PV) thinking for Rossby waves and related shear flow phenomena, this work develops a buoyancy-vorticity formulation of gravity waves in stratified shear flow, for which the nonlocality enters in the same way as it does for barotropic/baroclinic shear flows. This formulation provides a time integration scheme which is analogous to the time integration of the quasi-geostrophic equations with two, rather than one, prognostic equations, and a diagnostic equation for streamfunction through a vorticity inversion.

The invertibility of vorticity allows us to develop a gravity wave kernel view which provides a mechanistic rationalization of many aspects of the linear dynamics of stratified shear flow. The resulting kernel formulation is similar to the Rossby based one obtained for barotropic and baroclinic instability, however since there are two independent variables - vorticity and buoyancy - there are also two independent kernels at each level. Though having two kernels complicates the picture, the kernels are constructed so that they do not interact with each other at a given level.

1 Introduction

Stably-stratified shear flows support two types of waves and associated instabilities - Rossby waves which are related to horizontal potential vorticity (PV) gradients, and gravity waves which are related to vertical density gradients. Each of these wave types is associated with its own form of shear instabilities. Rossby wave instabilities (baroclinic instability) arise when the PV gradients change sign (Charney and Stern, 1962), while gravity wave related instabilities, of the type described for example by the Taylor-Goldstein equation, arise in the presence of vertical shear, when the Richardson number at some place becomes less than a quarter (Drazin and Reid, 1981). These conditions are quite easily obtained from the equations governing each type of instability, but their physical basis is much less clear. We do not have an intuitive understanding as we do, for example for convective instability which arises when the stratification itself is unstable (e.g. Rayleigh-Bernard, Rayleigh Taylor instabilities).

There are two main attempts to physically understand shear instabilities, which have gone a long way towards building a mechanistic picture. Over-reflection theory (e.g. Lindzen, 1988) explains perturbation growth in terms of an overreflection of waves in the cross-shear direction, off of a critical level region. Under the right flow geometry, overreflected waves can be reflected back constructively to yield normal mode growth, somewhat akin to a laser growth mechanism. Since this theory is based on quite general wave properties, it deals both with gravity wave and vorticity wave instabilities. A very different approach, which has been developed for Rossby waves, is based on the notion of Counter-propagating¹ Rossby Waves (CRWs, Bretherton, 1966; Hoskins et al., 1985). Viewed this way, instability arises from a mutual reinforcing and phase locking of such waves. This explicit formulation applies only to Rossby waves.

¹Since Rossby waves propagate in a specific direction, set by the sign of the mean PV gradient, their propagation is either with or counter the zonal mean flow.

Recently, the seemingly different Over-reflection and CRW approaches to shear instability have been united for the case of Rossby waves. A generalized form of CRW theory describes the perturbation evolution in terms of *kernel-wave* interactions. Defining a local vorticity anomaly, along with its induced meridional velocity, as a *kernel Rossby wave* (KRW), Heifetz and Methven (2005, hereafter HM) wrote the PV evolution equation in terms of mutual interactions between the KRWs, via a meridional advection of background PV, in a way which was mathematically similar to the classical CRW formulation. The KRWs are kernels to the dynamics in a way analogous to a Green function kernel. Harnik and Heifetz (2007, hereafter HH) used this kernel formulation to show that KRW interactions are at the heart of cross shear Rossby wave propagation and other basic components of Over-reflection theory, and in particular, they showed that overreflection can be explained as a mutual amplification of KRWs.

Given that CRW theory can explain the basic components of Rossby wave Over-reflection theory (wave propagation, evanescence, full, partial, and overreflection) using its own building blocks (KRWs), and Over-reflection theory, in turn, can rationalize gravity wave instabilities, it is natural to ask whether a wave-kernel interaction approach exists for gravity waves as well. Indeed, it has been shown that a mutual amplification of counter propagating waves applies also to the interaction of Rossby and gravity waves, or to mixed vorticity-gravity waves (Baines and Mitsudera, 1994; Sakai, 1989). These studies did not explicitly discuss the case of pure gravity waves, in the absence of background vorticity gradients. In this paper we present a general formulation of the dynamics of linear stratified shear flow anomalies in terms of a mutual interaction of analogous Kernel Gravity Waves (KGWs). We show how this formulation holds even when vorticity gradients, and hence “Rossby” type dynamics, are absent.

At first glance, Rossby waves and gravity waves seem to involve entirely different dynamics. The common framework used to describe Rossby waves is the quasi-geostrophic (QG) one, in

which motions are quasi-horizontal. In this framework, gravity waves, which are associated with the 'divergent' part of the flow, are filtered out. From the gravity wave perspective, however, the dynamics can be nondivergent, when viewed in three dimensions. Moreover, as we show later on, gravity waves involve vorticity dynamics, and this part of the dynamics has the same action-at-a distance features which quasi geostrophic dynamics has.

This paper presents a vorticity-buoyancy view of gravity waves, examines how this interplay between vorticity and buoyancy affects the evolution of stratified shear flow anomalies, and explores its use as a basis for a kernel view. Our general motivation in developing a vorticity-based kernel view of the dynamics is quite basic - in a similar way in which the KRW perspective has yielded mechanistic understanding for barotropic and baroclinic shear flows we expect that finding the corresponding gravity wave building blocks will provide a new fundamental insight into a variety of stratified shear flow phenomena, such as a basic mechanistic (rather than mathematical) understanding of the cross shear propagation of gravity wave signals, the necessary conditions for instability, the overreflection mechanism and the nonmodal growth processes in energy and enstrophy norms.

The paper is structured as follows. After formulating the simplified stratified shear flow equations in terms of vorticity-buoyancy dynamics (section 2), we examine how the interaction between these two fields is manifest in normal modes in general (section 3.1), in pure plane waves (section 3.2) and in a single interface between constant buoyancy and vorticity regions (section 3.3). We then go on to develop a kernel framework in section 4, for a single interface (section 4.1), two and multiple interfaces (sections 4.2 and 4.3) and the continuous limit (section 4.4). We discuss the results and summarize in section 5.

2 General formulation

We consider an inviscid, incompressible, Boussinesq, 2-D flow in the zonal - vertical ($x - z$) plane, with a zonally uniform basic state which varies with height and is in hydrostatic balance.

We start with the momentum and continuity equations, linearized around this basic state:

$$\frac{Du}{Dt} = -w\bar{U}_z - \frac{1}{\rho_0} \frac{\partial p}{\partial x} \quad (1a)$$

$$\frac{Dw}{Dt} = b - \frac{1}{\rho_0} \frac{\partial p}{\partial z} \quad (1b)$$

$$\frac{Db}{Dt} = -wN^2 \quad (1c)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad (1d)$$

where $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \bar{U} \frac{\partial}{\partial x}$, $\mathbf{u} = (u, w)$ is the perturbation velocity vector and its components in the zonal and vertical directions, respectively, \bar{U} and \bar{U}_z are the zonal mean flow and its vertical shear, p is perturbation pressure, ρ_o is a constant reference density, $b \equiv -\frac{\rho}{\rho_o}g$ is the perturbation buoyancy and $N^2 \equiv -\frac{g}{\rho_o} \frac{\partial \bar{\rho}}{\partial z}$ is the mean flow Brunt-Väisälä frequency, with ρ and $\bar{\rho}$ the perturbation and mean flow density, respectively, and g is gravity. We note that $N^2 = \bar{b}_z$, and use this in further notation.

We now take the curl of equations 1a, 1b, assume that all anomalies arise purely from an advection of the basic state, and express the buoyancy perturbation in terms of the vertical displacement ζ (not to be confused with the vertical component of vorticity). This yields an alternative set of equations for the vorticity and displacement tendencies:

$$\frac{Dq}{Dt} = -w\bar{q}_z - \bar{b}_z \frac{\partial \zeta}{\partial x} \quad (2a)$$

$$\frac{D\zeta}{Dt} = w \quad (2b)$$

$$b = -\bar{b}_z \zeta \quad (2c)$$

where $q = \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}$ and $\bar{q}_z = -\overline{U_{zz}}$.² Expressed in this way, the dynamic evolution of anomalies is reduced to a vorticity equation and a trivial kinematic relation for particle displacements, with vertical velocity mediating between the vorticity and displacement fields. We see that vorticity evolves either by a vertical advection of background vorticity gradients, or via a displacement (buoyancy) term. The latter expresses the tendency of material surfaces (which are surfaces of constant density) to flatten out, creating vorticity in the process. This is shown schematically for a sinusoidal perturbation of an interface between two layers of constant density, in figure 2. Concentrating on the zero point at which the surface tilts upward to the east ($\zeta_x > 0$), the tendency of gravity to flatten the interface will raise the surface just west of this point and sink it on its east, resulting in clockwise motion of the surface at the zero point, and a corresponding production of negative vorticity.

We further note that for nondivergent flow, a knowledge of w is sufficient to determine the entire flow field, which is directly related to q via a standard vorticity inversion. The following picture emerges: given vorticity and displacement perturbations, we can determine the vertical velocity associated with the vorticity perturbation. We note that the vertical velocity is non local, in the sense that it depends on the entire vorticity anomaly field. Given the vertical

²In a 3-D system, the component of vorticity perpendicular to the $x - z$ plane would be defined as minus our definition. We chose this convention to highlight the analogy to the case of horizontal barotropic flow.

velocity field, it will locally determine how the displacement field evolves, and along with the displacement field, will also determine how the vorticity field evolves. This scheme of inverting the vorticity field to get a vertical velocity (via a streamfunction), then using this velocity to time integrate the vorticity and displacement fields, inverting the new vorticity to get the new velocity, and so on, is similar in principle to the forward integration of the QG equations, only we now have two, instead of one dynamical variable.

Phrased this way, nonlocality enters the dynamics in the same way as it does in QG, and in the CRW formulation. We will use this to develop analogous Kernel Gravity Waves (KGWs), and a corresponding view of gravity wave dynamics.

More explicitly, introducing a streamfunction ψ , so that $u = -\frac{\partial\psi}{\partial z}$, $w = \frac{\partial\psi}{\partial x}$ and $q = \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = \nabla^2\psi$, and for a single zonal Fourier component with wavenumber k of the form e^{ikx} , $w = ik\psi$, and $q = -k^2\psi + \frac{\partial^2\psi}{\partial z^2}$. We can then express the inversion of q into w via a Green function formulation:

$$w(z) = \int_{z'} q(z') G(z', z, k) dz' \quad (3)$$

where $-k^2 G + \frac{\partial^2 G}{\partial z^2} = ik\delta(z - z')$. $G(z', z, k)$ depends on the boundary conditions and on the zonal wavenumber. For example, for open flow, which we assume here for simplicity, the Green function is:

$$G(z', z) = -\frac{i}{2} e^{-k|z-z'|} \quad (4)$$

Other boundary conditions will change $G(z', z, k)$ (see HM for some explicit examples) but will not affect the main points of this paper. Substituting 3 into equations 2a and 2b, using 4, we get the following set of equations which describe the evolution of perturbations of zonal wavenumber k :

$$\frac{Dq}{Dt} = \frac{i}{2} \bar{q}_z \int_{z'} q(z') e^{-k|z-z'|} dz' - ik\bar{b}_z \zeta \quad (5a)$$

$$\frac{D\zeta}{Dt} = -\frac{i}{2} \int_{z'} q(z') e^{-k|z-z'|} dz' \quad (5b)$$

For practical purposes eq. 5a, 5b can be discretized into N layers to obtain a numerical scheme (Appendix-A)

3 The buoyancy-vorticity interaction in normal modes

The interaction between the vorticity and buoyancy fields has to have specific characteristics for normal modes, which by definition, have a spatial structure which is fixed in time. Understanding how the two fields arrange themselves so that they evolve in concert yields some understanding of the mechanistic of gravity waves and their propagation, as well related instabilities. Moreover, the kernel view which we present in section 4 is based on the normal mode solutions of a single interface.

3.1 General background flow

We first note a few general properties of the normal mode solutions to equations 2a, 2b (or 5a, 5b). Multiplying equation 2b by \bar{q}_z , and adding to equation 2a, to get rid of w , we get an equation which involves only q and ζ :

$$\frac{D}{Dt} (q + \bar{q}_z \zeta) = -\bar{b}_z \frac{\partial \zeta}{\partial x} \quad (6)$$

We assume a normal mode solution of the form $e^{i(kx-ct)}$, where the zonal phase speed is allowed to be complex, $c = c_r + ic_i$, and we write the vorticity and displacement in terms of an amplitude and phase, as follows: $q = Qe^{i\alpha}$, $\zeta = Ze^{i\theta}$. Equating the real and imaginary parts of 6 gives:

$$\tan(\theta - \alpha) = \frac{\bar{b}_z c_i}{\bar{b}_z [c_r - \bar{U}] - \bar{q}_z ([c_r - \bar{U}]^2 + c_i^2)} \quad (7a)$$

$$\left(\frac{Q}{Z}\right)^2 = \frac{\left[\left([c_r - \bar{U}]\bar{q}_z - \bar{b}_z\right)^2 + c_i^2 \bar{q}_z^2\right]^2}{\left[\bar{b}_z[c_r - \bar{U}] - \bar{q}_z\left([c_r - \bar{U}]^2 + c_i^2\right)\right]^2 + \bar{b}_z^2 c_i^2} \quad (7b)$$

We see that for neutral normal modes ($c_i = 0$), q and ζ are either in phase or anti-phase, while for growing/decaying normal modes ($c_i \neq 0$) q and ζ have a different phase relationship, which, for a given complex zonal phase speed, varies spatially with \bar{U} , \bar{b}_z , and \bar{q}_z . Moreover, if there are no vorticity gradients ($\bar{q}_z = 0$), q and ζ have to be in quadrature at the critical surface (where $c_r = \bar{U}$).

3.2 Plane waves

The most common text book example of gravity waves is that of neutral infinite plane waves of the form:

$$q = q_0 e^{i(kx+mz-\omega t)} \quad ; \quad \zeta = \zeta_0 e^{i(kx+mz-\omega t)} \quad (8)$$

In this subsection we examine the vorticity-buoyancy interplay for such waves. Plugging the plane wave vorticity anomalies field (Eq. 8) into the integrals on the RHS of equations 5a, 5b), assuming a constant infinite basic state, yields the following equations for q and ζ (where we have used relation A.5):

$$(\bar{U} - c)q = \frac{\bar{q}_z}{K^2}q - \bar{b}_z\zeta \quad (9a)$$

$$(\bar{U} - c)\zeta = -\frac{q}{K^2} \quad (9b)$$

where $K^2 = k^2 + m^2$, is the total square wavenumber. This indeed yields neutral q and ζ structures, which propagate with the following (flow-relative) phase speed:

$$c^\pm - \bar{U} = -\frac{\bar{q}_z}{2K^2} \pm \sqrt{\left(\frac{\bar{q}_z}{2K^2}\right)^2 + \frac{\bar{b}_z}{K^2}} \quad (10)$$

We also obtain the following $\zeta - q$ relation:

$$q^\pm = [K^2(c^\pm - \bar{U})]\zeta^\pm \quad (11)$$

Note that $(c^\pm - \bar{U})$ is either positive or negative, and ζ and q are either in phase or anti-phase. The \pm superscript thus denotes the sign of $c^\pm - \bar{U}$, and the sign of the correlation between q and ζ . The results are also consistent with eq. 7a, 7b (with $c_i = 0$).

We note that for pure plane wave solutions, we need to assume \bar{q}_z and \bar{U} are both constant. To allow an analogy with single interface solutions of the next section, we assume both \bar{U} and \bar{q}_z are nonzero, though strictly speaking this can only be valid if there is an external vorticity gradient analogous to the QG β (i.e. that $\bar{q}_z = \bar{U}_{zz} + \Gamma$). Since no such Γ exists for the vertical direction (though this might be relevant for plasma flow with a background magnetic field vertical gradient), this derivation should only be treated as a thought experiment, done to get at the essential mechanistic of vorticity-buoyancy interplay in the presence of buoyancy and vorticity gradients.

Under the right conditions, Eq. 10 reduces to the well known Rossby and Gravity plane wave dispersion relations. When $\bar{b}_z = 0$ we get a pure vertical Rossby plane wave³:

$$(c^- - \bar{U})_{Rossby} = -\frac{\bar{q}_z}{K^2} \quad (12a)$$

where the dispersion relation is equivalent to the horizontal Rossby (1939) β plane wave (we ignored the redundant meaningless null solution). When $\bar{q}_z = 0$ the two pure Gravity plane waves are obtained:

$$(c^\pm - \bar{U})_{Gravity} = \pm \frac{\sqrt{\bar{b}_z}}{K} \quad (12b)$$

³We are using the term Rossby waves in its most general sense of vorticity waves on a vorticity gradient, even though Rossby waves generally refer to PV perturbations on a meridional PV gradient, while here we have vorticity anomalies on a vertical vorticity gradient.

3.3 Single interface solutions

To understand how the buoyancy and vorticity fields interact and evolve in normal modes, and how their dynamics affects their phase speed, it is illuminating to examine the simple case of an interface between two constant vorticity and buoyancy regions. The solution to this problem was already obtained by Baines and Mitsudera (1994). Nonetheless, we present it here to emphasize the vorticity-buoyancy interaction mechanistics. Moreover, the normal mode solutions to the single interface will serve as the basis for a kernel view of stratified shear flow anomalies (section 4), as was done for Rossby waves.

We examine the evolution of waves on a fluid with two regions of constant vorticity and buoyancy separated at an interface at $z = z_0$. The vorticity and buoyancy gradients are then:

$$\bar{q}_z = \Delta\bar{q}\delta(z - z_0) \quad (13a)$$

$$\bar{b}_z = \Delta\bar{b}\delta(z - z_0) \quad (13b)$$

where $\Delta\bar{q} = \bar{q}(z > z_0) - \bar{q}(z < z_0)$, and $\Delta\bar{b} = \bar{b}(z > z_0) - \bar{b}(z < z_0)$. Equation 2c implies that $b = \tilde{b}\delta(z - z_0)$ for this basic state, so that $\tilde{b} = -\Delta\bar{b}\zeta$. Note that b is proportional to minus the density, so that a stable stratification implies positive \bar{b}_z , and an upward interface displacement is associated with a positive density anomaly, or a negative buoyancy anomaly. Similarly, from equation 2a, we see that $q = \tilde{q}\delta(z - z_0)$. The δ -function form of q and b makes sense since a vertical displacement will induce anomalies only at the interface⁴. Note however, that even in the absence of vorticity gradients ($\bar{q}_z = 0$), a delta function in vorticity will be produced, due to the jump in buoyancy. In any case, the vertical velocity for this anomaly is simply $w = -\frac{i}{2}\tilde{q}$ (c.f. equations 3, 4).

⁴Note that we are ignoring the possibility of neutral normal modes associated with the continuous spectrum, for which which $q \neq 0$ and $b \neq 0$ in the $\bar{q}_z = 0$, $\bar{b}_z = 0$ regions.

Using the above definitions, and assuming normal mode wave solutions of the form $e^{ik(x-ct)}$, equations 5a,b yield the following dispersion relation (c.f Baines and Mitsudera, 1994):

$$c^\pm - \bar{U} = -\frac{\Delta\bar{q}}{4k} \pm \sqrt{\left(\frac{\Delta\bar{q}}{4k}\right)^2 + \frac{\Delta\bar{b}}{2k}} \quad (14)$$

We see that there are two normal mode solutions, one eastward and one westward propagating, relative to the mean flow, denoted by the \pm superscript, respectively. The dispersion relation reduces to the single background gradient solutions as expected: plugging $\bar{q}_z = 0$ yields the well known gravity wave dispersion relation for the pair of gravity wave normal modes (to be found in standard text books, e.g. Batchelor, 1980):

$$c^\pm - \bar{U} = \pm \sqrt{\frac{\Delta\bar{b}}{2k}} \quad (15)$$

whereas plugging $\bar{b}_z = 0$ we get the Rossby wave dispersion relation at a single interface (e.g. Swanson et al., 1997):

$$c - \bar{U} = -\frac{\Delta\bar{q}}{2k} \quad (16)$$

and another null solution $c = \bar{U}$ in which all fields are zero. Note that when $\bar{b}_z = 0$, the buoyancy perturbation is zero, and equation 2a yields the relation $q = -\bar{q}_z\zeta$, which is similar to the buoyancy-displacement relation in the presence of a density gradient⁵. The fact that q and ζ are directly tied this way means there is only one independent dynamical equation and one normal mode solution.

Coming back to the general case of nonzero \bar{b}_z and \bar{q}_z , it is easy to verify from the dispersion relation 14 that when viewed from the frame of reference moving with the mean flow, the westward mode propagates faster than the pure gravity wave while the eastward mode propagates slower than the pure gravity wave. This makes sense, apriori, since the additional Rossby type dynamics tends to shift the pattern westward.

⁵Equations 7a, 7b also yield this relation for $\bar{b}_z = 0$.

We can obtain a more mechanistic understanding by examining the normal mode structure. Since the two variables evolve differently – q evolves as a result of horizontal interface slopes and mean vorticity advection by the vertical velocity anomaly, while ζ evolves directly from the vertical velocity, which is tied by definition to the vorticity – the normal modes have to assume a specific structure which allows them to propagate in concert. This structure can be obtained from equation 5b and the definition of w :

$$\tilde{q}^\pm = 2k(c^\pm - \bar{U})\zeta^\pm \quad (17)$$

Note that q and ζ are either in phase or in anti-phase (as expected from equation 7a), depending on the phase propagation direction. Note also that equation 17 is consistent with equations 7a, 7b, with $c_i = 0$.

It is easiest to understand mechanistically how these modes propagate by first considering the pure vorticity and pure density jumps, and then their combination. The wave on a pure vorticity jump (illustrated in figure 1) propagates via the well known Rossby (1939) mechanism: the vertical velocity is always shifted a quarter wavelength to the east of the vorticity field (figure 1). For a positive \bar{q}_z , this tends to shift the pattern westward, as a result of the vertical advection of the background vorticity. Note that since q and w are in quadrature, the wave cannot change its own amplitude.

Looking at the + wave on a pure density jump (illustrated in figure 2a), the vertical velocity is shifted a quarter wavelength to the east of ζ^+ (since q and ζ are in phase) so that w is upward to the east of the interface ridges. This induces a positive displacement anomaly to the east, resulting in an eastward shifting of the displacement pattern relative to the mean flow. At the same time, the west-east interface slope ($\frac{\partial \zeta}{\partial x}$) is shifted a quarter wavelength to the west of the vorticity pattern, which, according to equation 2a (with $\bar{q}_z = 0$), will increase q to the east of the positive q center, and decrease it to the west, resulting as well in an eastward shifting of the

pattern. For the negatively correlated mode (ζ^-), the displacement remains the same but the vorticity, and associated vertical velocity fields flip signs. As a result, the wave pattern will be coherently shifted westward rather than eastward (see figure 2b).

For the combined vorticity-buoyancy jump (figure 3), the phase relations between q , ζ^\pm and w are the same as in the pure buoyancy jump case, but now, the vertical velocity also produces vorticity via advection of \bar{q}_z . The direct effect of this is only on the vorticity evolution. Looking for example at ζ^+ (figure 3a), the w - ζ phase relation will shift ζ^+ to the east. At the same time, the vorticity field evolves via two processes, according to equation 2a. The advection of background vorticity gradient by the vertical velocity tends to decrease \bar{q} to the east of its positive peak (the interface crest), but the interface slope tends to increase the vorticity there. Thus, the vertical velocity tends to shift the pattern westward (the Rossby wave mechanism) while the ζ gradient tends to shift it eastward (the gravity wave mechanism). Since the normal mode solution requires the vorticity and displacement fields to evolve coherently, the buoyancy gradient effect must win. For a given background flow, and a given wave-like interface displacement, this will only happen if the vertical velocity (and hence the vorticity anomaly) will be small enough. This constraint on the size of the w anomaly, per given unit ζ anomaly, further implies that the phase speed is small, since from equation 2b we have:

$$c - \bar{U} = \frac{i}{k} \frac{w}{\zeta} \quad (18)$$

where we should note that in our case iw/ζ is a real number, since w and ζ are $\pm\pi/2$ out of phase (in fact, growth requires w and ζ to not be in quadrature).

Similar arguments hold for the negatively correlated kernel (ζ^-), but now the vertical velocity is shifted $\pi/2$ to the west of the crests, so that the crests are shifted westward (figure 3b). At the same time the vertical advection of vorticity now works with the buoyancy gradient to create a negative \bar{q} anomaly to the west of the crest, resulting as well in a westward shift of the vorticity

pattern. Correspondingly, the resulting vorticity anomaly is larger than in the ζ^+ case, hence w/ζ and the phase speed are larger (in absolute value).

Equation 18 for the phase speed expresses a kinematic relation. Equation 17 further incorporates the relation between w and \tilde{q} assuming 2-D nondivergent flow and open boundary conditions ($w = -\frac{i}{2}\tilde{q}$). Both equations hold for a general interface. The basic state structure, characterized by \bar{q}_z and \bar{b}_z , affects the normal mode structure and dispersion relation via the vorticity equation, which determines the vorticity-per-displacement “production ratio”. For the pure buoyancy jump case, this production ratio is equal for the two modes, resulting in a symmetric dispersion relation. For the pure vorticity jump case, we can only have a negatively signed production ratio for a positive vorticity gradient and hence only westward propagating waves⁶.

4 A kernel view

The CRW view of shear instability, as a mutual amplification and phase locking of vorticity waves, was first formalized for the idealized case of two interfaces with oppositely signed vorticity jumps Bretherton (1966), and the instability was rationalized in terms of the interaction between anomalies on the two interfaces. For this case, each interface on its own supports a neutral normal mode, but when both interfaces exist, interaction between the waves may yield instability. Davies and Bishop (1994) formalized this for the Eady model, by mathematically expressing how the cross-shear velocity induced by a wave at one interface affects the phase and amplitude of the wave at the other interface, via its advection of the basic state vorticity. Unstable normal mode solutions then emerge when the two waves manage to resist the shear and phase lock in a configuration which also results in a mutual amplification.

⁶As opposed to \bar{b}_z , that must be positive in stably stratified flow, \bar{q}_z can be either positive or negative. In general, the Rossby wave propagation is to the left of the mean vorticity gradient, hence for negative \bar{q}_z we obtain a positive $c^+ - \bar{U}$ and a positive $\zeta - q$ correlation.

This approach also holds for more than two jumps, in which case the flow field evolution is expressed as a multiple wave interaction, as follows. A pure vorticity jump (lets say at $z = z_0$) supports a Rossby wave for which q and w are in quadrature, and since vorticity in this case is only affected by vorticity advection, this wave only propagates zonally (its phase changes) but it does not amplify. Since there are no vorticity gradients except at the jump, it does not create vorticity anomalies elsewhere. If, on the other hand, the basic state has additional vorticity jumps, new vorticity anomalies will be created at these jumps by the far-field vertical velocity of the $q(z_0)$ vorticity anomaly. These new anomalies, will in turn induce a far field w , which is not necessarily in quadrature with $q(z_0)$, allowing a change in amplitude, as well as phase. The full flow evolution is thus viewed as an interaction of multiple single-interface *kernels*, where by *kernel* we mean the delta function of vorticity at a given interface along with the vertical velocity which it induces. In other words, the kernel at z_0 is the normal mode solution of a mean flow which has a single vorticity jump at z_0 .

This view is useful because it can be generalized to continuous basic states by taking the limit of an infinite number of jumps spaced at infinitesimal intervals, as formulated in HM. So far, besides rationalizing shear instability, a kernel-based view of Rossby waves has yielded physical insight into basic phenomena like cross shear wave propagation, wave evanescence and reflection (HH), and various aspects related to non-normal growth (HM, de Vries and Opsteegh, 2006; de Vries and Opsteegh, 2007a,b; Morgan, 2001; Morgan and Chen, 2002).

With the goal of developing similar insight into gravity wave phenomena, we wish to develop a *kernel gravity wave* (KGW) formulation of stratified shear flow anomalies. Following the Rossby wave example, we start with examining the evolution of anomalies on multiple basic state jumps (both vorticity and buoyancy). We define our kernels to be the normal mode solutions to an isolated buoyancy-vorticity jump, then examine the evolution and mutual interaction of these

kernels on two and multiple jumps, and finally we take the continuous limit.

4.1 The general time evolution on a single jump

To understand the evolution of an arbitrary perturbation field as a multiple-kernel interaction, we need to first understand the evolution on a single interface. In the case of a pure vorticity jump, there is no buoyancy anomaly and there is a single normal mode solution (Eq 16)⁷, and correspondingly a single vorticity kernel. A density jump, on the other hand, supports two normal modes (Eq. 15, 17) at each interface. The $\zeta - \tilde{q}$ structure of the normal modes is determined by the requirement that their spatial structure not change with time. Since vorticity and buoyancy are independent variables (they influence each other's evolution but any combination of the two can exist), two modes are needed to represent the full anomaly field at a given level.

As with the vorticity case, we define the kernels to be the normal modes of a single jump, thus at each level there are two kernels, and each of these kernels has a specific $\zeta - \tilde{q}$ structure. Taking this a step further, an arbitrary $\zeta - \tilde{q}$ configuration can be uniquely divided into the two kernels, ζ^+ and ζ^- , as follows (using the same notation as for the normal modes):

$$\tilde{q} \equiv \tilde{q}^+ + \tilde{q}^- = 2k \left[(c^+ - \bar{U})\zeta^+ + (c^- - \bar{U})\zeta^- \right] \quad (19a)$$

$$\zeta = \zeta^+ + \zeta^- \quad (19b)$$

where we have used the relation 17. The two kernel components are then obtained from \tilde{q} and ζ (see appendix B):

$$\zeta^+ = \frac{1}{2k[(c^+ - \bar{U}) - (c^- - \bar{U})]} \left[-2k(c^- - \bar{U})\zeta + \tilde{q} \right] \quad (20a)$$

⁷Again, we do not consider the normal modes associated with the continuous spectrum.

$$\zeta^- = -\frac{1}{2k[(c^+ - \bar{U}) - (c^- - \bar{U})]} [-2k(c^+ - \bar{U})\zeta + \tilde{q}] \quad (20b)$$

In the simplified case of a pure buoyancy jump, we note that $(c^+ - \bar{U}) = -(c^- - \bar{U})$, and these equations reduce to:

$$\zeta^+ = \frac{1}{2} \left(\zeta + \frac{\tilde{q}}{2k(c^+ - \bar{U})} \right) \quad (21a)$$

$$\zeta^- = \frac{1}{2} \left(\zeta + \frac{\tilde{q}}{2k(c^- - \bar{U})} \right) \quad (21b)$$

Once the projection onto the two kernels is found, the temporal evolution is readily obtained since each kernel propagates at its own phase speed, that corresponding to the single jump normal mode, and there is no interaction between the kernels. This is stated mathematically in appendix B, using matrix notation, with the two kernels being the eigenvectors of the propagator matrix for the combined vorticity-displacement vector, and the phase speeds are directly related to the eigenvalues.

4.2 Two interfaces

The simplest case which allows for kernel interactions is a mean flow with two interfaces. Such a setup was examined by Baines and Mitsudera (1994). Here we formulate the problem in terms of kernel interactions, for the most general case of buoyancy and vorticity jumps. We denote the interface locations by z_1 and z_2 , their distance by $z_2 - z_1 = \Delta z > 0$, and the vorticity and buoyancy jumps by $\Delta\bar{q}_{1/2}$ and $\Delta\bar{b}_{1/2}$ where the subscript 1/2 denotes the interface. The four kernels (two for each interface) are thus denoted by $\zeta_{1/2}^\pm$. The dynamics is then described in terms of the evolution of the kernels, as follows. Writing equations 5a, 5b for the two interfaces we get a 4×4 matrix equation:

$$\frac{\partial}{\partial t} \boldsymbol{\eta} = \mathbf{A} \boldsymbol{\eta} \quad (22a)$$

where

$$\boldsymbol{\eta} = \begin{pmatrix} \tilde{q}_1 \\ \zeta_1 \\ \tilde{q}_2 \\ \zeta_2 \end{pmatrix} \quad (22b)$$

and \mathbf{A} is defined in Appendix-C. Using 20a, 20b, we can write the transformation

$$\boldsymbol{\zeta} = \mathbf{T}\boldsymbol{\eta} \quad (23a)$$

where

$$\boldsymbol{\zeta} = \begin{pmatrix} \zeta_1^+ \\ \zeta_1^- \\ \zeta_2^+ \\ \zeta_2^- \end{pmatrix} \quad (23b)$$

and the transformation matrix \mathbf{T} is defined in Appendix-C. Using the similarity transformation, we get the dynamic equation for the kernel displacements, written in matrix and compact form, respectively:

$$\frac{\partial}{\partial t}\boldsymbol{\zeta} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\boldsymbol{\zeta} \quad (24a)$$

$$\dot{\zeta}_{1/2}^\pm = -ik \left[(c^\pm \zeta^\pm)_{1/2} \pm e^{-k\Delta z} \left(\frac{c^\pm - \bar{U}}{c^+ - c^-} \right)_{1/2} \left[(c^+ - \bar{U})\zeta^+ + (c^- - \bar{U})\zeta^- \right]_{2/1} \right] \quad (24b)$$

where 1/2 indicate the interface and \pm indicate the kernel, and we have used equation 14 for c_\pm , to cancel out some terms in the derivation. Note that the expression in the inner squared brackets has a flipped 2/1 index, indicating the contribution of the opposite boundary. Note also that

$$(c^+ - c^-)_{1/2} = \left[\left(\frac{\Delta \bar{q}}{2k} \right)^2 + 2 \frac{\Delta \bar{b}}{k} \right]_{1/2}^{\frac{1}{2}} \quad (25)$$

Equation 24b shows, as expected, that in the absence of interaction between the interfaces, $\frac{\partial}{\partial t}\zeta_{1/2}^\pm = -ik (c^\pm \zeta^\pm)_{1/2}$, which is simply the two kernel propagation equations. This confirms that the two kernels at a given interface do not interact at all (i.e. a + kernel can only interact with the + and - kernels of the other interface).

Written also in terms of displacement amplitudes (Z) and phases (ϵ):

$$\zeta_{1/2}^{\pm} = Z_{1/2}^{\pm} e^{i\epsilon_{1/2}^{\pm}} \quad (26)$$

and taking the real and imaginary parts of equation 24b yields the amplitude and phase evolution equations:

$$\begin{aligned} \dot{Z}_{1/2}^{\pm} = \pm k e^{-k\Delta z} \left(\frac{c^{\pm} - \bar{U}}{c^{+} - c^{-}} \right)_{1/2} \times \\ \left[\left[(c^{+} - \bar{U}) Z^{+} \right]_{2/1} \sin(\epsilon_{2/1}^{+} - \epsilon_{1/2}^{\pm}) + \left[(c^{-} - \bar{U}) Z^{-} \right]_{2/1} \sin(\epsilon_{2/1}^{-} - \epsilon_{1/2}^{\pm}) \right] \end{aligned} \quad (27a)$$

$$\begin{aligned} -\frac{1}{k} \dot{\epsilon}_{1/2}^{\pm} = c_{1/2}^{\pm} \pm \frac{e^{-k\Delta z}}{Z_{1/2}^{\pm}} \left(\frac{c^{\pm} - \bar{U}}{c^{+} - c^{-}} \right)_{1/2} \times \\ \left[\left[(c^{+} - \bar{U}) Z^{+} \right]_{2/1} \cos(\epsilon_{2/1}^{+} - \epsilon_{1/2}^{\pm}) + \left[(c^{-} - \bar{U}) Z^{-} \right]_{2/1} \cos(\epsilon_{2/1}^{-} - \epsilon_{1/2}^{\pm}) \right] \end{aligned} \quad (27b)$$

As expected, equation 27a indicates that each mixed buoyancy-vorticity kernel on its own is neutral. Its amplitude can grow only due to advection of mean buoyancy and vorticity gradients by the vertical velocity induced by the kernels of the opposite interface. This advecting velocity is proportional to the inducing kernel displacement amplitudes and the nature of the inversion between vertical velocity and displacement. Hence it decays with the distance between the interfaces (indicated by the Green function). Only the components of the inducing velocity which are in phase with the induced kernel's amplitude yield growth (the sines on the RHS result from the fact that the displacement and vertical velocity of a kernel are $\pi/2$ out of phase). Equation 27b indicates that the instantaneous phase speed of each kernel, $-\dot{\epsilon}/k$, is comprised of its natural speed without interaction (equation 14), and an effect of the opposing interface kernels. The latter only arises from that part of the induced vertical velocity which is in phase with the kernel (and hence the cosines on the RHS).

A normal mode solution of the full two-interface system requires phase locking of the four kernels, which is obtained when the RHS of 27b is the same and constant for the four kernels (this constant is the real normal mode phase speed c_r). Normal mode solutions should also have a constant growth rate (kc_i) for the four kernels. This is obtained only if $\dot{Z}_{1/2}^\pm/Z_{1/2}^\pm = kc_i$ in 27a. In a companion paper we analyze the normal mode dispersion relation and structure of pure gravity waves in terms of this mutual phase locking interaction for the two interface problem Umurhan et al. (2007). We note that although this interaction is more complex for gravity waves than for Rossby waves because there are two kernels at each interface, vertical shear will discriminate between these kernels, because phase locking dominantly occurs for kernels which counter propagate with respect to the mean flow Umurhan et al. (2007).

4.3 Multiple interfaces

As a step towards a continuous formulation, we generalize the 2-interface equations to an arbitrary number of jumps:

$$\hat{\zeta}_j^\pm = -ik \left[\left(c^\pm \hat{\zeta}^\pm \right)_j \pm \left(\frac{c^\pm - \bar{U}}{c^+ - c^-} \right)_j \sum_{n=1, \neq j}^N e^{-k|j-n|\Delta z} \left[(c^+ - \bar{U})\zeta^+ + (c^- - \bar{U})\zeta^- \right]_n \right] \quad (28)$$

Decomposing $\hat{\zeta}$ into amplitude and phase, as in 26 yields

$$\hat{Z}_j^\pm = \pm k \left(\frac{c^\pm - \bar{U}}{c^+ - c^-} \right)_j \sum_{n=1, \neq j}^N e^{-k|j-n|\Delta z} \left(\left[(c^+ - \bar{U})Z^+ \right]_n \sin(\epsilon_n^+ - \epsilon_j^\pm) + \left[(c^- - \bar{U})Z^- \right]_n \sin(\epsilon_n^- - \epsilon_j^\pm) \right) \quad (29a)$$

$$-\frac{1}{k}\hat{\epsilon}_j^\pm = c_j^\pm \pm \left(\frac{c^\pm - \bar{U}}{c^+ - c^-} \right)_j \sum_{n=1, \neq j}^N \frac{e^{-k|j-n|\Delta z}}{Z_j^\pm} \left(\left[(c^+ - \bar{U})Z^+ \right]_n \cos(\epsilon_n^+ - \epsilon_j^\pm) + \left[(c^- - \bar{U})Z^- \right]_n \cos(\epsilon_n^- - \epsilon_j^\pm) \right) \quad (29b)$$

4.4 The continuous limit

Our overall goal is to develop a kernel formulation which holds for continuous basic states, as was done for Rossby waves (HM). Apparently, this simply entails taking the limit of equation 28, for $\Delta z \rightarrow 0$ with $N \rightarrow \infty$. However, this limit is ill defined when $\bar{b}_z \neq 0$. To see this, we note that equation 28 involves writing equations 5a and 5b in terms of the kernels ζ^+ , ζ^- . This requires dividing the vorticity perturbation to the two kernels, according to equation 19a, using the dispersion relation 14, and taking the above limit. Writing $\Delta b = \bar{b}_z \Delta z$, $\Delta q = \bar{q}_z \Delta z$, and noting that the single interface vorticity \tilde{q} is the amplitude of the vorticity delta function (which has units of vorticity times length) hence $\tilde{q} = q \Delta z$, we get:

$$q = \left[-\frac{\bar{q}_z}{2} \pm \sqrt{\left(\frac{\bar{q}_z}{2}\right)^2 + 2k \frac{\bar{b}_z}{\Delta z}} \right] \zeta^\pm \quad (30)$$

which blows up in the limit of $\Delta z \rightarrow 0$, for $\bar{b}_z \neq 0$. When $\bar{b}_z = 0$, this limit is well defined, and yields the basic relation $q = -\bar{q}_z \zeta$. Indeed, for the Rossby wave case, the kernel formulation works for the continuous limit (HM). When $\bar{b}_z \neq 0$, the formulation, in its current form, fails because an infinitesimal-width q , which has a finite value (as opposed to a δ function q , which is infinitesimal in width and infinite in amplitude), has a negligible contribution to w , and hence to the generation of ζ c.f. equation 5b. This sets the ratio ζ^\pm/q to zero. Unlike the single interface case, where ζ directly influences the time tendency of q (equation 5a), and q directly influences the evolution of ζ (equation 5b), here only ζ produces q , so that q and ζ cannot propagate in concert to form a normal-mode kernel structure.

Because, however, the vorticity inversion is such that its action-at-a-distance decays with distance, the vertical velocity at a given location is most strongly influenced by the local vorticity anomaly. In other words, even though the contribution of $q(z)$ to $w(z)$ is infinitesimal, it is larger than the contribution of q at any other height. Mathematically, we can get around this problem

by simply looking at the anomaly averaged over a finite interval of width Δz .

To do this we discretize the domain as in Appendix-A, using equations A.1 for the basic state gradients, and define the average of any perturbation quantity f on height z_j as

$$\hat{f}_j \equiv \frac{1}{\Delta z} \int_{z_j - \Delta z/2}^{z_j + \Delta z/2} f dz \quad (31)$$

The vertical velocity at z_j induced by the vorticity perturbation located between $(z_j - \Delta z/2, z_j + \Delta z/2)$ and approximated by \hat{q}_j , for small Δz is

$$\hat{w}_j = -\frac{i}{2} \hat{q}_j \int_{z_j - \Delta z/2}^{z_j + \Delta z/2} e^{-k|z-z'|} dz' \approx -\frac{i}{2} \hat{q}_j \Delta z \quad (32)$$

Substituting 32 in 2b and seeking wavelike solutions yields

$$\Delta z \hat{q}_j = [2k(c^\pm - \bar{U}_j)] \hat{\zeta}_j^\pm \quad (33)$$

which converges to 17 for $\hat{q}_j = \tilde{q} \delta(z - z_j)$ and $\Delta z \rightarrow 0$. In order to get an expression for $(c^\pm - \bar{U}_j)$, we take the second total time derivative of equation 2a:

$$\frac{D^2 \hat{q}_j}{Dt^2} = -\bar{q}_{zj} \frac{D \hat{w}_j}{Dt} - ik \bar{b}_{zj} \hat{w}_j \quad (34)$$

For the chunk formulation this yields the following dispersion relation:

$$c^\pm - \bar{U}_j = -\frac{\gamma \bar{q}_{zj}}{4k^2} \pm \sqrt{\left(\frac{\gamma \bar{q}_{zj}}{4k^2}\right)^2 + \frac{\gamma \bar{b}_{zj}}{2k^2}} \quad (35)$$

where the nondimensionalized number $\gamma \equiv k \Delta z$ should be taken small enough (for relation 32 to hold). It is straightforward to verify that for a buoyancy-vorticity staircase profile, this expression converges to 14 as $\gamma \rightarrow 0$. Substituting 35 into equation 33 we obtain the local $\hat{\zeta} - \hat{q}$ relation:

$$\bar{b}_z \hat{\zeta}_j^\pm = \pm \left[\left(\frac{\gamma \bar{q}_{zj}}{4k^2}\right)^2 + \frac{\gamma \bar{b}_{zj}}{2k^2} \right]^{\frac{1}{2}} \hat{q}_j \quad (36)$$

Using this relation, and noting that

$$(c^+ - c^-)_j = \left[\left(\frac{\gamma \bar{q}_{zj}}{2k^2} \right)^2 + 2 \frac{\gamma \bar{b}_{zj}}{k^2} \right]^{\frac{1}{2}} \quad (37)$$

we get kernel amplitude and phase equations which are exactly similar to 29a, 29b, with $\Delta b = \frac{\bar{b}_z}{k} \gamma$, $\Delta q = \frac{\bar{q}_z}{k} \gamma$, and q and ζ being the chunk-averaged fields \hat{q} and $\hat{\zeta}$.

We can now use these equations to numerically study the evolution of buoyancy and vorticity anomalies on an arbitrary continuous profile in terms of kernel interactions. This formulation, like the piecewise basic state one, does not converge for $\gamma \rightarrow 0$. This lack of convergence stems from the fact that when the chunk size shrinks to zero, the vertical velocity induced by the chunk vanishes (equation 32), and no displacement is induced. However, when we look at a finite region, the mechanistic picture of buoyancy-vorticity interactions, with two chunk-kernels at each discrete (finite width) level, holds. In particular, we can examine the case of constant \bar{b}_z and \bar{q}_z , with a pure plane wave solution of the form A.3, calculate \hat{q} and $\hat{\zeta}$, and the corresponding $\hat{\zeta}^+$ and $\hat{\zeta}^-$ (c.f. equation 20a, 20b), and plug into the chunk version of equation 29a and 29b. Equation 29a does indeed yield zero amplitude growth while equation 29b yields the discretized version of the plane wave dispersion relation 10 using equation A.4 for the sum terms (not shown).

5 Summary and discussion

PV thinking, which includes the concepts of PV inversion and action-at-a-distance, provides a mechanistic understanding of many aspects of barotropic and baroclinic shear flow anomalies (Hoskins et al., 1985). In this paper we present a corresponding vorticity viewpoint which is appropriate for gravity waves and stratified shear flow dynamics.

We show that vorticity dynamics is central to stratified shear flow anomalies, which are comprised of a continuous interaction between vorticity and buoyancy perturbations. Horizon-

tal buoyancy gradients generate vorticity locally, whereas vorticity induces a nonlocal vertical velocity, which, in turn, generates both fields by advecting the background buoyancy and vorticity. Thus, the interplay between vorticity and buoyancy anomalies fully describes the linear dynamics on stratified shear flows, both modal and nonmodal. Casting the dynamics in this way essentially provides us with a time integration scheme which is analogous to the time integration of QG equations, with two rather than one prognostic equations, and a diagnostic equation for streamfunction which is related to vorticity as in QG.

Since the buoyancy anomalies arise purely from vertical advection of the background buoyancy field, we can express the buoyancy anomalies via the local vertical displacement, and the dynamics reduce to a mutual interaction between vorticity and displacement fields. The vertical displacement equation is then the simple statement that vertical velocity is the time derivative of displacement. For normal modes, the two fields (vorticity and displacement) assume specific phase and amplitude relations. In particular, for neutral modes, vorticity and displacement are either in phase or in anti-phase with respect to each other.

For baroclinic/barotropic flow, PV invertibility allows us to develop a kernel view of the dynamics, where anomalies are comprised of localized Kernel Rossby Waves, which interact with each other. This mutual interaction occurs via the nonlocal flow induced by each kernel's PV anomaly. In analogy, our buoyancy-vorticity view also allows a clear separation between local and non local dynamics, with the nonlocal part of the dynamics stemming from the vorticity inversion. This makes a kernel view of the dynamics possible, and a major part of this paper deals with developing it.

As was done for vorticity kernels, we define our kernels based on single interface vorticity-buoyancy normal modes. Since both vorticity and buoyancy are involved in stratified shear flow, there are two normal modes at each level, and thus, two kernels. These are comprised

of an eastward propagating wave with positively correlated vorticity and displacement, and a westward propagating wave with negatively correlated vorticity and displacement. Though having two kernels complicates the picture, the kernels are constructed so that they do not interact with each other at a given level.

An arbitrary initial vorticity-displacement distribution can, at each level, be uniquely decomposed into the two kernel components, each of which evolves according to its own internal dynamics, with an influence from kernels at other levels (via their induced vertical velocities). The dynamic equations for vorticity and displacement can thus be decomposed and written in terms of the evolution of the two kernels (at each level).

To get a sense of multiple kernel interactions between many levels, we consider a flow with multiple buoyancy-vorticity jumps, which we initially perturb at a single interface located at $z = z_0$. The resulting vorticity anomaly at z_0 will induce a far-field vertical velocity, which will perturb the other jump interfaces. The resulting displacements of each interface are associated with buoyancy and vorticity anomalies. Note that for the case of a pure buoyancy jump, an initial interface distortion can only create a buoyancy anomaly, and no vorticity anomaly, because the vorticity is constant across the jump. The anomalies initially induced by an interface distortion evolve according to the local vorticity-displacement interaction dynamics. Though these initial ζ and q perturbations are initially in phase, their amplitude ratio is not necessarily the normal mode one. Thus, both normal modes will be excited, and will proceed to propagate in opposite directions independently of each other. These new excited kernels will induce a far field w , which will affect other interfaces. The full flow evolution can be viewed as an interaction of multiple pairs of single-interface kernels.

We note, however, that the description of the evolution in terms of a superposition of the two kernels whose density and vorticity anomalies are either in or out of phase is not necessarily

the most intuitive one. For example, for a pure buoyancy interface, an initial displacement-only anomaly will excite the two modes with equal amplitude (ζ^+ and ζ^- are in phase but q^+ and q^- are in anti phase). Such a superposition of the two kernels with equal amplitude actually gives rise to a standing wave (in the moving frame of the mean flow) in which the vorticity and the density anomalies are in quadrature (see appendix B). We expect the kernel view to be the most intuitive in many cases (as we show in a companion paper, Umurhan et al., 2007), while in other setups, an alternative standing-oscillation building block might provide a more intuitive description of the dynamics (currently under investigation).

To make the kernel formulation applicable to general stratified shear flow profiles, we use the multiple-jump formulation as a basis for a continuous flow. The continuous limit has been found to be quite subtle. Our kernel formulation is based on the assumption that clear vorticity-displacement relations exist, which are used to define our kernels. For the pure Rossby wave case, where the piecewise setup converges nicely to the continuous problem, vorticity and displacement are directly related ($q = -\bar{q}_z \zeta$). When buoyancy gradients and corresponding buoyancy anomalies exist, the strict relation between q and ζ exists for jumps, but becomes singular in the continuous limit. This stems from the fact that an important part of this strict $\zeta - q$ relation is the generation of ζ by q via its induced vertical velocity. In the continuous limit, an infinitesimal localized q generates an infinitesimal w , and hence an infinitesimal ζ . In order to recover a $\zeta - q$ relation, we recover a locally induced w , by considering finite-width vorticity chunks, which generate a small, but nonzero ζ . This chunk formulation can be viewed as a numerical discretization approximation of the full $\zeta - q$ dynamic equations.

To summarize, we present a vorticity-buoyancy framework for stratified shear flow anomalies, which yields new insight into the mechanistic of gravity waves. This framework can now be applied to gain a mechanistic understanding of a variety of stratified shear flow phenomena. In

a companion paper, Umurhan et al. (2007), we examine instabilities on a constant shear flow with two density jumps, using the above buoyancy-vorticity kernel view. Even though in this system there are no vorticity gradients, and hence no vorticity waves, instability arises out of a mutual interaction and phase locking of counter propagating gravity waves, in analogy to the CRW view of classical models like those of Rayleigh and Eady. Since much of the kernel dynamics found for Rossby waves is also present in Umurhan et al. (2007), we expect the kernel view to yield a mechanistic understanding of other aspects of stratified shear flow, like nonmodal growth of anomalies, and basic processes like cross shear gravity wave propagation, reflection, and overreflection. One of the central theorems of stratified shear flows is the Miles-Howard criterion for instability - that the Richardson number (R_i) be smaller than $1/4$ somewhere in the domain (Miles, 1961; Howard, 1961). Though it has been arrived at in various ways, our mechanistic (as opposed to mathematical) understanding of it is only partial. Some mechanistic understanding is gained in the context of both overreflection and the counter-propagating wave interaction frameworks, as follows. $R_i < 1/4$ is also a criterion for wave evanescence in the region of a wave critical surface, which allows significant transmission of gravity waves through the critical level (Booker and Bretherton, 1967, see also Van Duin and Kelder, 1982, where this is explicitly demonstrated for the far field). On the one hand, the existence of an evanescent region is a necessary component for overreflection (e.g Lindzen, 1988). On the other hand, as pointed out by Baines and Mitsudera (1994), this evanescent region serves to divide the fluid into two wave regions, whose waves can interact and mutually amplify. This suggests the two approaches to gravity wave instabilities, based on overreflection and on counter-propagating wave-interaction, may be related in a similar way to what was shown for Rossby waves using the kernel approach (HH). Our gravity wave kernel formalism is one step towards understanding this relation, and in the end we expect the Richardson number criterion to become much clearer.

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Appendix - A

An explicit numerical scheme for time integration of equation 5

We consider a smooth basic state with vorticity $\bar{q}(z)$ and buoyancy $\bar{b}(z)$, which we discretize into N layers with a vertical width Δz so that $z_n = n\Delta z$ and $n = 1, 2, \dots, N$ (in principle Δz can be a function of z as well, depending on the complexity of the profiles, but for simplicity we choose it constant). The discretized mean vorticity and buoyancy gradients are then:

$$\bar{q}_{z_n} \equiv \frac{\bar{q}(z_{n+1/2}) - \bar{q}(z_{n-1/2})}{\Delta z}; \quad \bar{b}_{z_n} \equiv \frac{\bar{b}(z_{n+1/2}) - \bar{b}(z_{n-1/2})}{\Delta z} \quad (\text{A.1a,b})$$

The discrete form of equations 5a,b is then:

$$\dot{q}_j = -ik(q_j + \bar{b}_{z_j}\zeta_j) + \frac{i}{2}\bar{q}_{z_j} \sum_{n=1}^N q_n e^{-k|j-n|\Delta z} \Delta z \quad (\text{A.2a})$$

$$\dot{\zeta}_j = -ik\zeta_j - \frac{i}{2} \sum_{n=1}^N q_n e^{-k|j-n|\Delta z} \Delta z \quad (\text{A.2b})$$

The plane wave dispersion relation,10 can be recovered, for instance, when we take constant values of $\bar{q}_{z_n} = \bar{q}_z$ and $\bar{b}_{z_n} = \bar{b}_z$, assume a discretized plane wave solution of the form:

$$\begin{pmatrix} q_n \\ \zeta_n \end{pmatrix} = \begin{pmatrix} q_0 \\ \zeta_0 \end{pmatrix} e^{i(kx + mn\Delta z - \omega t)} \quad (\text{A.3})$$

and recall that in the limit $N \rightarrow \infty$ & $\Delta z \rightarrow 0$ we get:

$$\sum_{n=1}^N e^{(imn - k|j-n|\Delta z)} \Delta z = \sum_{n=-N/2}^{N/2} e^{(imn - k|j-n|\Delta z)} \Delta z \longrightarrow \frac{2k}{K^2} e^{imz_j} \quad (\text{A.4})$$

This converges to the following integral relation which was used to derive equation 3.2 by plugging the plane wave vorticity anomalies field (Eq. 8) into the integrals on the RHS of equations 5a, 5b), assuming a constant infinite basic state:

$$\int_{z'=-\infty}^{\infty} q(z') e^{-k|z-z'|} dz' = q_0 e^{i(kx - \omega t)} \int_{z'=-\infty}^{\infty} e^{imz'} e^{-k|z-z'|} dz' = \frac{2k}{K^2} q_0 e^{i(kx + mz - \omega t)} \quad (\text{A.5})$$

Appendix - B

Eigenvalue representation of the single interface dynamics

For a single interface equations 5a,b can be written in matrix form:

$$\frac{\partial}{\partial t} \begin{pmatrix} \tilde{q} \\ \zeta \end{pmatrix} = -i \begin{pmatrix} k\bar{U} - \frac{\Delta\tilde{q}}{2} & k\Delta\bar{b} \\ \frac{1}{2} & k\bar{U} \end{pmatrix} \begin{pmatrix} \tilde{q} \\ \zeta \end{pmatrix} \quad (\text{B.1})$$

whose solution is

$$\begin{pmatrix} \tilde{q} \\ \zeta \end{pmatrix} = \zeta_0^+ \begin{pmatrix} 2k(c^+ - \bar{U}) \\ 1 \end{pmatrix} e^{-ikc^+t} + \zeta_0^- \begin{pmatrix} 2k(c^- - \bar{U}) \\ 1 \end{pmatrix} e^{-ikc^-t} \quad (\text{B.2})$$

where c^\pm is defined by equation 14. Hence, at $t = 0$:

$$\begin{pmatrix} \zeta_0^+ \\ \zeta_0^- \end{pmatrix} = \begin{pmatrix} 2k(c^+ - \bar{U}) & 2k(c^- - \bar{U}) \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \tilde{q}_0 \\ \zeta_0 \end{pmatrix} = \frac{1}{2k[(c^+ - \bar{U}) - (c^- - \bar{U})]} \begin{pmatrix} 1 & -2k(c^- - \bar{U}) \\ -1 & 2k(c^+ - \bar{U}) \end{pmatrix} \begin{pmatrix} \tilde{q}_0 \\ \zeta_0 \end{pmatrix} \quad (\text{B.3})$$

in agreement with equations 20a, 20b.

Note that for the case of a pure density jump, an initial displacement-only anomaly ($\tilde{q}_0 = 0$), which is what we get in response to an induced w , will excite the two normal modes with equal amplitude: $\zeta_0^+ = \zeta_0^- = \frac{1}{2}\zeta_0$ (c.f. B.3). Using (B.2) and 15, this gives a standing wave (in the frame of reference of the mean flow), for which \tilde{q} and ζ are actually in quadrature.

$$\tilde{q} = \zeta_0 \sqrt{2k\Delta b} \sin \left(\sqrt{\frac{k\Delta b}{2}} t \right) \sin [k(x - \bar{U}t)] \quad (\text{B.4a})$$

$$\zeta = \zeta_0 \cos \left(\sqrt{\frac{k\Delta b}{2}} t \right) \cos [k(x - \bar{U}t)] \quad (\text{B.4b})$$

Appendix - C

Derivation of equation 20

The matrix \mathbf{A} is obtained from direct substitution in equations 5a,b (recall that $\int_{z'} q(z')e^{-k|z-z'|}dz' = \tilde{q}_j e^{-k|z-z_j|}$ for $q = \tilde{q}_j \delta(z - z_j)$):

$$\mathbf{A} = -i \begin{pmatrix} k\bar{U}_1 - \frac{\Delta\bar{q}_1}{2} & k\Delta\bar{b}_1 & -\frac{\Delta\bar{q}_1}{2}e^{-k\Delta z} & 0 \\ \frac{1}{2} & k\bar{U}_1 & \frac{1}{2}e^{-k\Delta z} & 0 \\ -\frac{\Delta\bar{q}_2}{2}e^{-k\Delta z} & 0 & k\bar{U}_2 - \frac{\Delta\bar{q}_2}{2} & k\Delta\bar{b}_2 \\ \frac{1}{2}e^{-k\Delta z} & 0 & \frac{1}{2} & k\bar{U}_2 \end{pmatrix} \quad (\text{C.1})$$

The $\tilde{q} - \zeta$ to $\zeta^+ - \zeta^-$ transformation matrix \mathbf{T} is obtained directly from 20a, 20b:

$$\mathbf{T} = \begin{pmatrix} \left[\frac{1}{2k[(c^+ - \bar{U}) - (c^- - \bar{U})]} \right]_1 & \left[-\frac{2k(c^- - \bar{U})}{2k[(c^+ - \bar{U}) - (c^- - \bar{U})]} \right]_1 & 0 & 0 \\ \left[-\frac{1}{2k[(c^+ - \bar{U}) - (c^- - \bar{U})]} \right]_1 & \left[\frac{2k(c^- - \bar{U})}{2k[(c^+ - \bar{U}) - (c^- - \bar{U})]} \right]_1 & 0 & 0 \\ 0 & 0 & \left[\frac{1}{2k[(c^+ - \bar{U}) - (c^- - \bar{U})]} \right]_2 & \left[-\frac{2k(c^- - \bar{U})}{2k[(c^+ - \bar{U}) - (c^- - \bar{U})]} \right]_2 \\ 0 & 0 & \left[-\frac{1}{2k[(c^+ - \bar{U}) - (c^- - \bar{U})]} \right]_2 & \left[\frac{2k(c^- - \bar{U})}{2k[(c^+ - \bar{U}) - (c^- - \bar{U})]} \right]_2 \end{pmatrix} \quad (\text{C.2})$$

The similarity transformation is thus:

$$\mathbf{T}^{-1} \mathbf{A} \mathbf{T} = -ik \times$$

$$\begin{pmatrix} c_1^+ & 0 & e^{-k\Delta z} \left(\frac{c^+ - \bar{U}}{c^+ - c^-} \right)_1 (c^+ - \bar{U})_2 & e^{-k\Delta z} \left(\frac{c^+ - \bar{U}}{c^+ - c^-} \right)_1 (c^- - \bar{U})_2 \\ 0 & c_1^- & -e^{-k\Delta z} \left(\frac{c^- - \bar{U}}{c^+ - c^-} \right)_1 (c^+ - \bar{U})_2 & -e^{-k\Delta z} \left(\frac{c^- - \bar{U}}{c^+ - c^-} \right)_1 (c^- - \bar{U})_2 \\ e^{-k\Delta z} \left(\frac{c^+ - \bar{U}}{c^+ - c^-} \right)_2 (c^+ - \bar{U})_1 & e^{-k\Delta z} \left(\frac{c^+ - \bar{U}}{c^+ - c^-} \right)_2 (c^- - \bar{U})_1 & c_2^+ & 0 \\ -e^{-k\Delta z} \left(\frac{c^- - \bar{U}}{c^+ - c^-} \right)_2 (c^+ - \bar{U})_1 & -e^{-k\Delta z} \left(\frac{c^- - \bar{U}}{c^+ - c^-} \right)_2 (c^- - \bar{U})_1 & 0 & c_2^- \end{pmatrix} \quad (\text{C.3})$$

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Figure Captions

Fig 1. Schematic illustration of the normal mode solution to a pure vorticity jump, on the zonal-vertical plane. The wavy line denotes the interface, the arrows denote the vertical velocity, and the sign of q and ζ at the crests is marked. The shaded regions on the top plot (at time $t = t_0$) denote regions of positive vorticity generation and negative displacement generation. This vorticity and displacement generation pattern shifts the wave westward relative to the mean flow (denoted by the thick horizontal arrow). The wave position at a quarter period later is shown below (marked $t = t_0 + \Delta t$). For this case there is only the negatively correlated ζ^- mode (see text for details), which is a westward propagating Rossby wave (for $\bar{q}_z > 0$).

Fig 2. Schematic illustration of the normal mode solutions to a pure buoyancy jump. Shown are (a) the negatively correlated, westward propagating ζ^- , and (b) the positively correlated, eastward propagating ζ^+ . The schematics are similar to figure 1, with the shaded regions denoting regions of negative displacement generation in both plots, and positive/negative vorticity generation in the top/bottom plots.

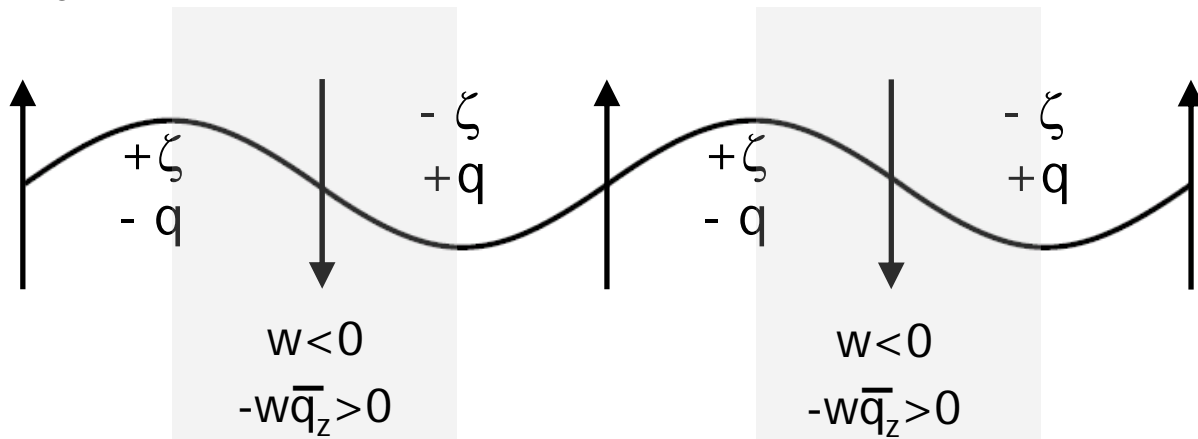
Fig 3. As in figure 2 but for the combined buoyancy-vorticity jump. Note that the phase speed, as well as vertical velocity arrows denote both the direction and relative magnitude, so that ζ^- propagates westward faster, and has a larger vertical velocity, than the pure buoyancy mode of figure 2, while ζ^+ propagates eastward slower, with a weaker vertical velocity (per unit displacement).

$$\bar{q}_z > 0$$

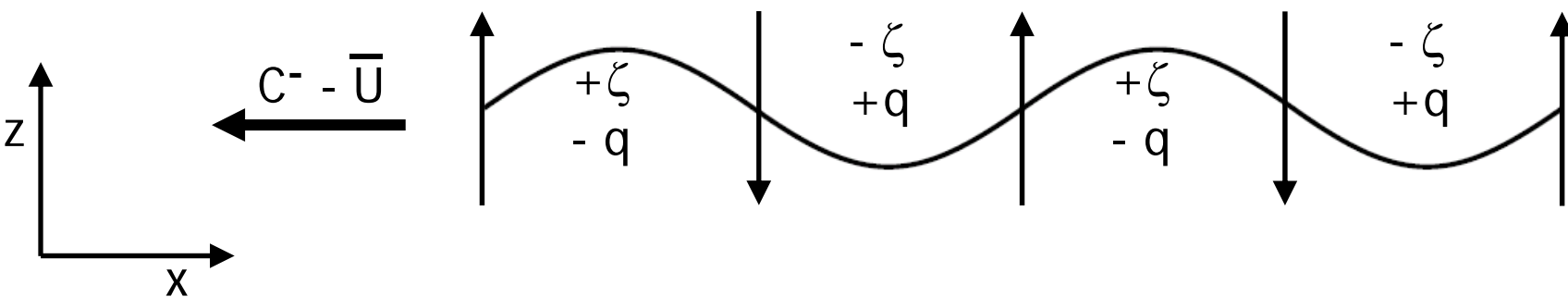
$$\frac{Dq}{Dt} = -w\bar{q}_z$$

$$\frac{D\zeta}{Dt} = w$$

$t = t_0$



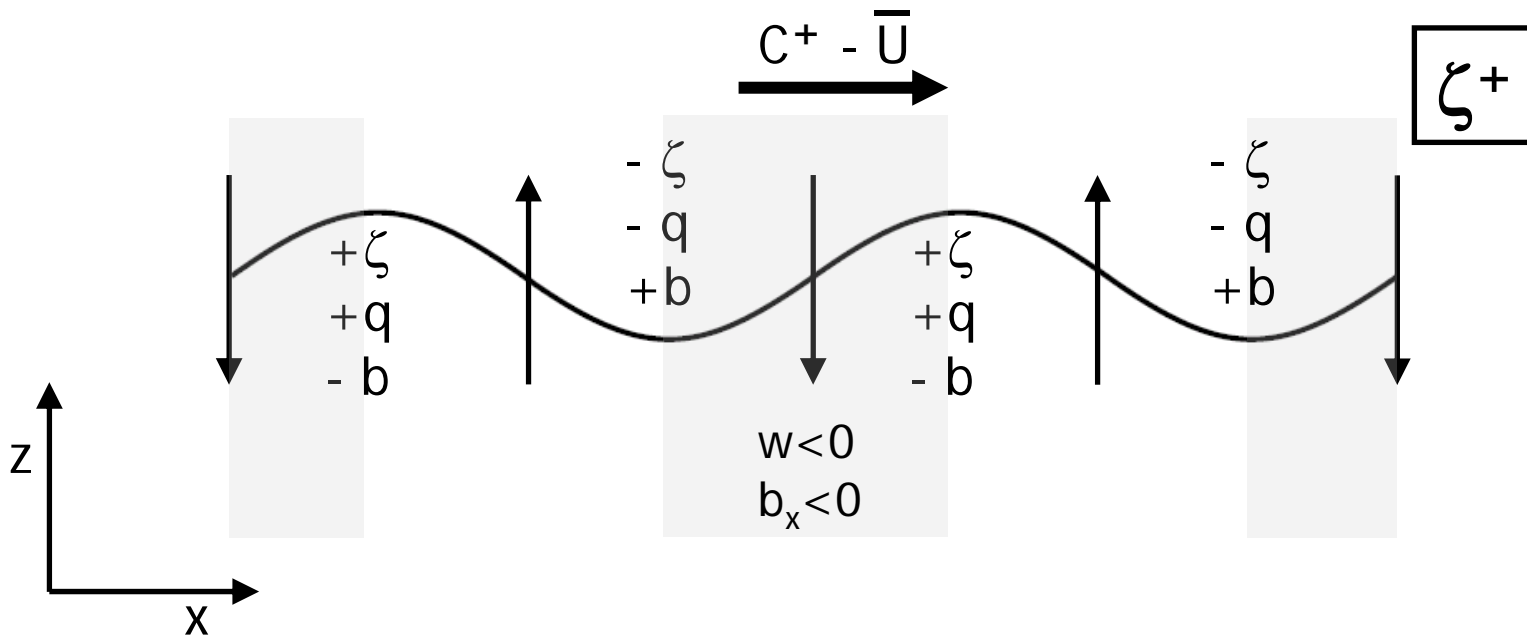
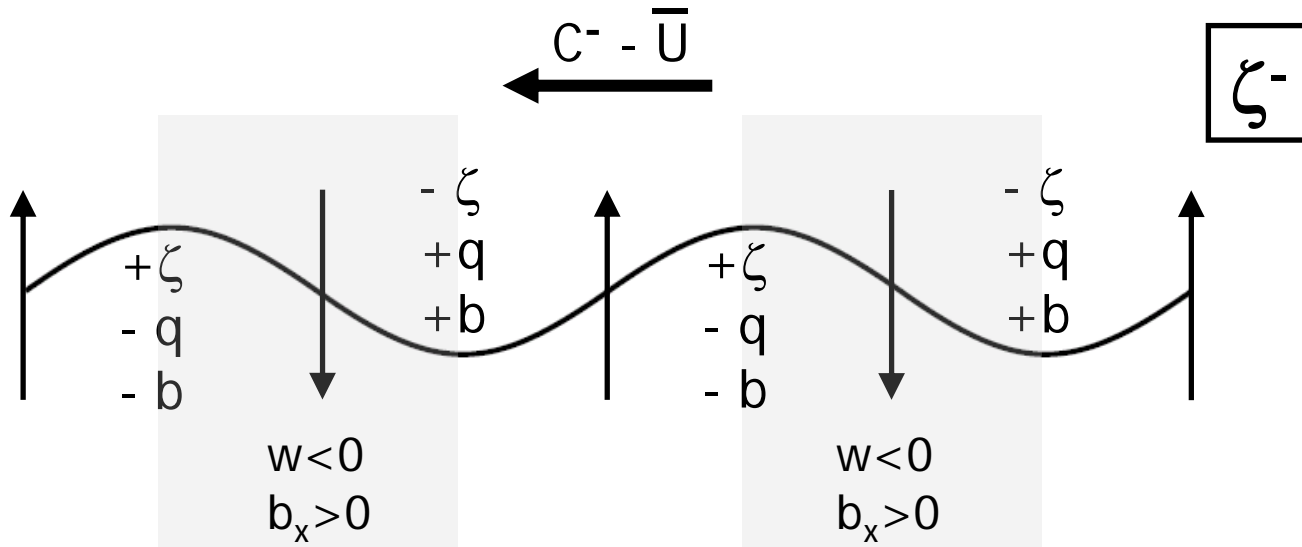
$t = t_0 + \Delta t$



$$\bar{b}_z > 0$$

$$\frac{Dq}{Dt} = \frac{\partial b}{\partial x}$$

$$\frac{D\zeta}{Dt} = w$$



$$\bar{q}_z > 0$$

$$\bar{b}_z > 0$$

$$\frac{Dq}{Dt} = -w\bar{q}_z + \frac{\partial b}{\partial x}$$

$$\frac{D\zeta}{Dt} = w$$

