Numerical modeling, Tutorial 2 : One dimensional advection of a tracer

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Let us consider a trace species whose concentration $q(x,t)$, in kg/kg for instance, depends on dimension x and time only, over a domain $[0,L]$ in x and $[0,T]$ in time. Let us assume that this trace species is advected by a non divergent U =cste wind field, with a constant mass of air per unit length (ρ in kg/m). You may choose $L = 1000$ km and $U = 10$ m s^{-1} to fix ideas or the legend of your plots, but those parameters are not important for the exercise.

1. We assume that at initial time, the tracer is distributed as a Gaussian function around $x_0 = L/5.$, i. e. $q(x,t = 0) = exp(-[(x-x_0)/\lambda]^2)$ with $\lambda = L/15$. What will be the tracer concentration at a time t after.

2. Given a discretization of 100 points in the domain $[0,L]$, write a fortran code which prints in column the values of x_i and $q_{i,0} = q(x_i, t = 0)$ for the 100 grid points. Plot these values on the screen. One can start from the program available at

   ```
   http://www.lmd.jussieu.fr/~hourdin/COURS/M2/TD/TD2maquette.f90
   ```

   or on

   ```
   ~/MON/TD/TD2maquette.f90
   ```

   In this simple case, one can easily compute the exact concentration at time t. Print the values at $t = 0.6 * L/U$ in a second file. Superimpose the two curves on a graph.

   Please parameterize the number of points in x, by declaring for instance an array for x_i such as:

   ```
   implicit none
   integer, parameter :: imax=100
   integer i

   real, dimension(imax) :: xxx
   do i=1,imax
      xxx(i)=...
   ```

3. Give the 3 advection schemes based on 3 finite difference estimates of the space derivative of the tracer concentration. We will call upstream, centered and downstream schemes the schemes centered respectively on a left, centered and right space derivative (for $U > 0$). Based on a Taylor expansion in the vicinity of point x_i, compare the accuracy of the various discretization schemes for the tracer concentration space derivative when δx goes to 0. Which scheme is the most accurate ?

4. Write a time loop so as to compute the tracer advection with an upstream scheme over 350 time steps, with $\delta t = 0.2 \times \delta x/U$. Be careful of well separating in the code: 1) the declarations of variables, then 2) the initialization (attribution of a value) and 3) the computation itself. Initialization must be done in particular before the time loop. $[(x-x_0)/y]^2 \rightarrow$ ((x-x0)/y))**2 in Fortran. Also note that you should not have more than one time loop in your code, with as many lines as you want in x inside.

   In order to record the results, one can write:

   ```
   character*4 file
   do it=1,nt
   ....
       if (mod(it,10)==1) then
          file="q..."
          write(file(2:4),'(i3.3)') it  ! writing, with 3 charcers, an intefer
                                        ! in position 2 to 4 of a
   ```

```
                                        ! 4-character string.
      open (10,file=file,form='formatted')
      do i=1,imax
         write(10,*) xxx(i),qqq(i)
      enddo
      close(10)
   endif
```

which will create a file each 10 time step, called for instance 'q091' at time step 91. Then curves can be plotted on the screen with the following command:

```
xmgrace q*
```

You can also send the image in a file directly with the line command

```
xmgrace -hardcopy -hdevice JPEG -printfile out.jpg q*
```

('q*' means that the command is applied for all the files starting by letter 'q' in your current directory, ie here : q001, q011, q021 ...). But you can use any other mean to plot the curves.

The coding can be facilitated by introducing two "arrays" to store the tracer concentrations q at time n and $n+1$, for instance 'qn(:)' and 'qnp1(:)'.

For the left boundary condition, one can assume $q = 0$ for $x < 0$.

5. Adapt the program to use a centered scheme in x. Compare the results of those two schemes. Comment in terms of accuracy, by looking in particular at the first time steps, and with the Taylor expansion done above.

   Comment on the more or less physical characteristics of the scheme.

6. Compute the advection with both schemes assuming that at initial time, the tracer is the combination of two sin modes, and of a "numerical oscillation" at the grid scale : $q_{i,1} = \sin(4\pi x_i/x_{\max}) + \sin(17\pi x_i/x_{\max}) + OSC(i)/2$ where $OSC(i) = 1$ for even and -1 for odd values of i, or, en fortran

```
pi=2.*asin(1.)
xxx(i)=i*deltax
qinitial(i)=sin(4*pi*xxx(i)/xmax)+sin(17*pi*xxx(i)/xmax)+0.5*(-1.+2.*mod(i,2))
```

   One may assume that flux entering on the left side or outgoing on the right side of the domain are nuls, or alternatively code periodic boundary conditions.

7. Show that the upstream scheme can be written as the sum of the centered scheme plus a term which can be interpreted as a finite difference formulation of a diffusion operator

$$\frac{\partial q}{\partial t} = K \frac{\partial^2 q}{\partial x^2} \qquad (1)$$

   Give the value of K. Apply the diffusion alone to the initial tracer field of the previous question.

8. Which scales dissipate first ? Comment with respect to the time evolution of the amplitude $A(t)$ of a sin mode $A(t)\sin(kx)$ solution of Eq **??**.

9. Rerun the computation of dissipation over 10 time steps, saving results at each individual time step, but with β=0.3 and 0.6, where $\beta = K\delta t/(\delta x)^2$. Analyse the results obtained considering a solution of the discretized scheme of the form $q_{i,n} = A_n OSC(i)$, where n is the time step (a link can be established with the results of tutorial #1).

Write a report on Tutorial 1 and 2.
Answer :
for TD1 : "Analysis"
for TD2 : questions 3 and 5 to 9.

Once your repport is completed, save it as a PDF file. Name it as `TD12_name1_name2.pdf`, `name1` being your name (very important for me not to mix the various files). Then send the pdf file by e-mail at `hourdin@lmd.jussieu.fr`. **put also your name and first name at the top of the document**.