Pergamon

J. Quant. Spectrosc. Radiat. Transfer Vol. 61, No. 4, pp. 433–441, 1999

Printed in Great Britain
PII: S0022-4073(97)00214-8
0022–4073/99 $—see front matter

INVERSE GAUSSIAN k-DISTRIBUTIONS

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(Received 3 July 1997)

Abstract—k-distributions corresponding to Malkmus' narrow band model are inverse Gaussian distributions. Inverse Gaussian theory developments are therefore directly relevant to gas radiative transfer modeling. The present text illustrates some significant benefits that could be made from this observation: (i) k-distribution formulations are simplified, (ii) numerical integration procedures can be optimized for each new configuration type, and (iii) frequently encountered integrals can be solved analytically and numerical integrations can be avoided. This last point is illustrated with the computation of infra-red cooling rates in planetary atmospheres. © 1999 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

Malkmus' narrow band model[1] has become a common tool for modeling gas radiation. It is a two parameter model for the average transmission function of a gas column of length l over a narrow spectral interval:

$$\bar{\tau}(l) = \frac{1}{\Delta\nu}\int_{\Delta\nu} \exp(-k_\nu l)\mathrm{d}\nu = \exp[\phi - \phi^*(l)] \tag{1}$$

with

$$\phi^*(l) = \phi(1 + 2\mu l/\phi)^{1/2}. \tag{2}$$

The two parameters are the average absorption coefficient μ and the shape parameter $\phi = 2\bar{\gamma}/\bar{\delta}$ where $\bar{\gamma}$ is the average half linewidth at half height and $\bar{\delta}$ is the average line to line spacing.

With the emergence of the k-distribution and correlated-k-distribution methods,[2] Malkmus' model has been frequentely used to derive a model for the probability density function f of the absorption coefficient k within a narrow spectral interval. This was performed via an inverse Laplace transform of the average transmission function:[3]

$$f(k) = L^{-1}[\bar{\tau}(l)] = \sqrt{\frac{\phi\mu}{2\pi k^3}}\exp\left[-\frac{\phi}{2}\frac{(k-\mu)^2}{\mu k}\right]. \tag{3}$$

The aim of the present text is to point out that $f(k)$ is an inverse Gaussian distribution and to suggest that direct benefits can be made from the abundant inverse Gaussian literature.

2. INVERSE GAUSSIAN DISTRIBUTIONS

Detailed descriptions of the inverse Gaussian theory and its related literature can be found in two recent monographs;[4,5] we summarize hereafter the aspects most relevant to the present application. All referred material may be found in these two references, except where specifically mentioned. The need for this univariate distribution appeared in the framework of Brownian motion theory. The inverse Gaussian was first introduced by both Schrodinger and Smoluchowski in 1915 (as the "first passage time" distribution of Brownian motion with drift) and the first important statistical

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properties were established by Tweedie in 1957. However, it is only recently that the inverse Gaussian distribution has been intensively used and analysed. Its main features are its ability to model highly skewed distributions and its numerous analogies with the normal distribution.

The probability density function $\mathfrak{I}$ of an inverse Gaussian distributed variable X is

$$pdf(x) = \mathfrak{I}(x; a, b) = \sqrt{\frac{ab}{2\pi x^3}} \exp\left[-\frac{b}{2}\frac{(x-a)^2}{ax} \right]. \tag{4}$$

The definition interval is $]0, +\infty[$ and a and b are positive parameters. The positive moments about zero $\mathscr{E}_r$ are given by the following expression:

$$\forall r > 0, \mathscr{E}_r(a, b) = E[X^r] = a^r \sum_{s=0}^{r-1} \frac{(r-1+s)!}{(r-1-s)!}(2b)^{-s} \tag{5}$$

and the negative moments are related to the positive ones via

$$\forall r \geq 0, \mathscr{E}_{-r}(a, b) = E[X^{-r}] = \frac{\mathscr{E}_{r+1}(a, b)}{a^{2r+1}}. \tag{6}$$

The parameter a is the distribution mean and the variance is $var(X) = a^2/b$. A useful property of inverse Gaussians is also

$$\mathfrak{I}(x; a, b) = a^{-1}\mathfrak{I}(x/a; 1, b). \tag{7}$$

This means that any inverse Gaussian distributed variable can be normalized and described with a one-parameter distribution called the standard Wald's distribution:

$$\mathscr{W}(x; b) = \mathfrak{I}(x; 1, b). \tag{8}$$

For this reason the parameter b is named the shape parameter. The cumulative distribution function of the inverse Gaussian was expressed by Shuster in terms of the standard normal distribution Γ:

$$\int_0^x \mathfrak{I}(u; a, b)\mathrm{d}u = \Gamma\left[-\sqrt{\frac{b}{x/a}}(1 - x/a) \right] + \mathrm{e}^{2b}\Gamma\left[-\sqrt{\frac{b}{x/a}}(1 + x/a) \right]. \tag{9}$$

Also worth mentioning here is the algorithm designed in 1976 by Michael and co-workers for generating random variates from an inverse Gaussian distribution. Sampling of X from the inverse Gaussian distribution with mean a and shape parameter b can be performed with a two steps algorithm. First a value x_1 is generated randomly according to a chi-square distribution with one degree of freedom. x_1 is used to compute x_2 such that

$$x_2 = a\left(1 + \frac{x_1}{2b} - \sqrt{\frac{x_1}{b} + \left(\frac{x_1}{2b}\right)^2} \right). \tag{10}$$

Then a Bernoulli trial is performed where the value $x = x_2$ is retained with probability $P = a/(a + x_2)$ and $x = a^2/x_2$ with probability $1 - P$.

The reciprocal of the inverse Gaussian has interesting properties that are relevant to gas radiation applications. If X is an inverse Gaussian distributed variable with mean a and shape parameter b, then the probability density function $\mathscr{R}$ of the random variable $Y = 1/X$ is

$$pdf(y) = \mathscr{R}(y; a, b) = ay\mathfrak{I}(y; 1/a, b). \tag{11}$$

Its moments can be obtained from those of X according to

$$E(Y^r) = \frac{E(X^{r+1})}{a^{2r+1}}. \tag{12}$$

Its cumulative distribution function is not given specific attention in the literature. The mathematical developments leading to the following expression are given in Appendix A:

$$\int_0^y \mathscr{R}(u; a, b)\mathrm{d}u = \Gamma\left[-\sqrt{\frac{b}{ay}}(1 - ay) \right] - \mathrm{e}^{2b}\Gamma\left[-\sqrt{\frac{b}{ay}}(1 + ay) \right]. \tag{13}$$

3. OPTIMISATION OF NUMERICAL INTEGRATION PROCEDURES

At the outset of the paper is the observation that the probability density function $f(k)$ corresponding to Malkmus' narrow band model is an inverse Gaussian distribution of mean μ and shape parameter ϕ. Equation (3) can be simply written as

$$f(k) = \Im(k; \mu, \phi). \tag{14}$$

The inverse Gaussian analytical properties are therefore useful when deriving k-distribution methods for spectral integration over narrow band intervals. Whatever the approach chosen, a central part of such methods is the numerical integration over the k-domain.[2,6–8] The spectral average quantities are expressed as

$$\bar{A} = \frac{1}{\Delta\nu}\int_{\Delta\nu} A(k_\nu)\,\mathrm{d}\nu = \int_0^\infty A(k) f(k)\,\mathrm{d}k \tag{15}$$

and k-integrals are commonly estimated with Gaussian quadratures or Monte Carlo methods. The optimization of such integration procedures require a confident knowledge of k-intervals that are most significant to the final integral result. This is very much dependent on the type of dominant radiative exchanges (surface–surface, gas–surface or gas–gas exchanges), the spectral structure (the shape parameter ϕ) and the optical depth ($\kappa = \mu l$). The net exchange formulations is practical in such contexts: radiative transfers are described via Net-Exchange Rates (NER) ψ between zones considered by pairs which allows to distinguish between different exchange types.[9] The monochromatic NER between two elementary surfaces, one elementary surface and one elementary gas volume or two elementary gas volumes are, respectively,

$$\frac{\partial^2\psi_\nu^{ss}}{\partial S_P \partial S_Q}(P, Q) = \frac{1}{l_{PQ}^2}[\mathbf{n}_S(P)\cdot\mathbf{n}_{PQ}][\mathbf{n}_S(Q)\cdot\mathbf{n}_{QP}][B_\nu(Q) - B_\nu(P)]\exp[-k_\nu l_{PQ}], \tag{16}$$

$$\frac{\partial^2\psi_\nu^{gs}}{\partial V_P \partial S_Q}(P, Q) = \frac{1}{l_{PQ}^2}[\mathbf{n}_S(Q)\cdot\mathbf{n}_{QP}][B_\nu(Q) - B_\nu(P)]k_\nu\exp[-k_\nu l_{PQ}], \tag{17}$$

$$\frac{\partial^2\psi_\nu^{gg}}{\partial V_P \partial V_Q}(P, Q) = \frac{1}{l_{PQ}^2}[B_\nu(Q) - B_\nu(P)]k_\nu^2\exp[-k_\nu l_{PQ}], \tag{18}$$

where l_{PQ} is the distance between the two points, $\mathbf{n}_{PQ}$ is the unit vector in direction $\mathbf{PQ}$, $\mathbf{n}_S(Q)$ is the unit vector normal to the surface in Q, B_ν is the spectral blackbody intensity and k_ν the spectral absorption coefficient. Equation (15) allows to express narrow band average net-exchange rates as

$$\frac{\partial^2\bar{\psi}^{ss}}{\partial S_P \partial S_Q}(P, Q) = \frac{1}{l_{PQ}^2}[\mathbf{n}_S(P)\cdot\mathbf{n}_{PQ}][\mathbf{n}_S(Q)\cdot\mathbf{n}_{QP}][\bar{B}(Q) - \bar{B}(P)]\bar{\tau}(l_{PQ})\int_0^\infty f^{ss}(k; l_{PQ})\,\mathrm{d}k, \tag{19}$$

$$\frac{\partial^2\bar{\psi}^{gs}}{\partial V_P \partial S_Q}(P, Q) = -\frac{1}{l_{PQ}^2}[\mathbf{n}_S(Q)\cdot\mathbf{n}_{QP}][\bar{B}(Q) - \bar{B}(P)]\frac{\partial\bar{\tau}}{\partial l}(l_{QP})\int_0^\infty f^{gs}(k; l_{PQ})\,\mathrm{d}k, \tag{20}$$

$$\frac{\partial^2\bar{\psi}^{gg}}{\partial V_P \partial V_Q}(P, Q) = \frac{1}{l_{PQ}^2}[\bar{B}(Q) - \bar{B}(P)]\frac{\partial^2\bar{\tau}}{\partial l^2}(l_{QP})\int_0^\infty f^{gg}(k; l_{PQ})\,\mathrm{d}k, \tag{21}$$

with

$$f^{ss}(k; l) = \frac{1}{\bar{\tau}(l)}\exp(-kl)f(k), \tag{22}$$

$$f^{gs}(k; l) = \frac{-1}{(\partial\bar{\tau}/\partial l)(l)}k\exp(-kl)f(k), \tag{23}$$

$$f^{gg}(k; l) = \frac{1}{(\partial^2\bar{\tau}/\partial l^2)(l)}k^2\exp(-kl)f(k). \tag{24}$$

The distributions f^{ss}, f^{gs} and f^{gg} are k-distributions over the $]0, +\infty[$ interval. They are to be interpreted as indicators of the contributions of k to the surface–surface, gas–surface and gas–gas

NER (respectively) at distance l. For instance, $f^{ss}(k; l_{PQ})\,dk$ is the fraction of the NER between two elementary surfaces at points P and Q that occurs at frequencies for which k_ν is in the interval $[k, k + dk]$:

$$f^{ss}(k; l_{PQ})\,dk = \frac{(\partial^2\psi_k^{ss}/\partial S_P \partial S_Q)(P, Q)\,dk}{(\partial^2\bar{\psi}^{ss}/\partial S_P \partial S_Q)(P, Q)} \tag{25}$$

These three distributions can be expressed conveniently in terms of inverse Gaussian distributions:

$$f^{ss}(k; l) = \mathfrak{I}(k; \mu^*(l), \phi^*(l)), \tag{26}$$

$$f^{gs}(k; l) = \mathscr{R}\left(k; \frac{1}{\mu^*(l)}, \phi^*(l)\right), \tag{27}$$

$$f^{gg}(k; l) = \mu^*(l)^{-2}\left(1 + \frac{1}{\phi^*(l)}\right)^{-1} k^2 \mathfrak{I}(k; \mu^*(l), \phi^*(l)). \tag{28}$$

with $\mu^*(l) = \mu(1 + 2\mu l/\phi)^{-1/2}$ and $\phi^*(l) = \phi(1 + 2\mu l/\phi)^{+1/2}$. For f^{ss}, the above expression is obtained by replacing $f(k)$ in Eq. (22) with its expression of Eq. (3), $\bar{\tau}(l)$ with its expression of Eq. (1) and by regrouping the exponentials. For f^{gs} and f^{gg}, Eqs. (22) and (26) are used as well as the two following relations deduced from Eq. (1):

$$\frac{\partial\bar{\tau}}{\partial l}(l) = -\mu^*(l)\bar{\tau}(l), \tag{29}$$

$$\frac{\partial^2\bar{\tau}}{\partial l^2}(l) = \bar{\tau}(l)\mu^*(l)^2\left(1 + \frac{1}{\phi^*(l)}\right). \tag{30}$$

The cumulative g^{ss} and g^{gs} of f^{ss} and f^{gs} can be derived analytically as direct consequences of Eqs. (9) and (13). The derivation of the cumulative g^{gg} of f^{gg} is detailed in Appendix A.

The fact that analytical expressions are available for these three functions allows easy quantifications, for a given spectral band, of the contributions of any k-interval to the radiative exchanges at distance l for the three possible exchange types. Apart from the interests in terms of physical interpretations, the fact that g^{ss}, g^{gs} and g^{gg} exist in close form permits the derivation of optimized k-interval discretizations. This was illustrated in Ref. 7 for g^{ss} (called the cumulative transmission function and denoted h) and can be extended to g^{gs} and g^{gg} for specific studies of gas–surface and gas–gas exchanges. These cumulative distributions are plotted in Figs. 1–3 for $\phi = 10^{-3}$ and two optical thicknesses $\kappa \in \{0.1, 10\}$, as functions of $g(k) = \int_0^k f(u)du$. Figure 1 illustrates that g^{ss} is a regular function of g. This fact ensures that the values that are likely in accordance with f correspond to values that are likely according to f^{ss} and therefore they are significant for the NER

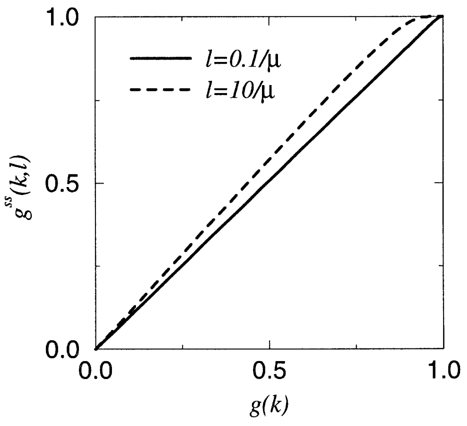

Fig. 1. k-cumulatives g^{ss} for surface–surface exchanges at distance l as a function of the inverse transmission cumulative ($g(k) = \int_0^k f(u)\,du$); $\phi = 10^{-3}$.

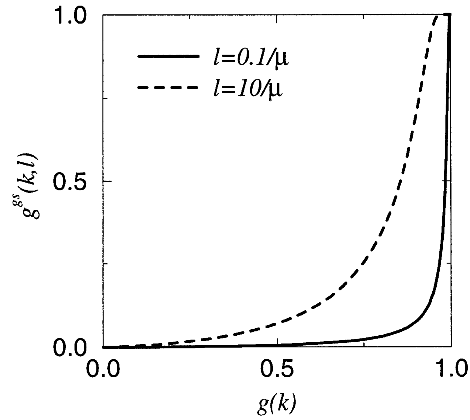

Fig. 2. k-cumulatives g^{gs} for gas–surface exchanges at distance l as a function of the inverse transmission cumulative $(g(k) = \int_0^k f(u)\,du)$; $\phi = 10^{-3}$.

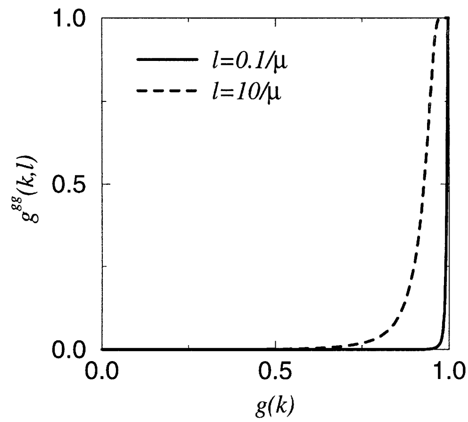

Fig. 3. k-cumulatives g^{gg} for gas–gas exchanges at distance l as a function of the inverse transmission cumulative $(g(k) = \int_0^k f(u)\,du)$; $\phi = 10^{-3}$.

integral (Eq. (20)). On the contrary, Figs. 2 and 3 show that g^{gs} and g^{gg} are strongly varying functions of g for g-values close to unity (i.e. for large k values). This means that a small interval of g has a large influence on the gas–surface and gas–gas NER integrals (Eqs. (20) and (21)). These observations imply that regular quadratures can be used for surface–surface exchanges and are coherent with the fact that most quadrature schemes give specific attention to the vicinity of $g = 1$ for gas–surface and gas–gas exchanges.[6–8] Note however that different quadrature refinements may be required depending on optical thickness.

A significant benefit that we made of the inverse Gaussian theory concerned k-sampling algorithms for Monte Carlo simulations.[10,11] Combinations of k-distribution models and Monte Carlo methods for spectral integration of radiative transfer equation require random generations of large samples of absorption coefficient values according to adapted k-distributions. First attempts were strongly limited (in terms of computing requirements) by the inefficiency of numerical k-sampling procedures because of the high skewness of k-distributions encountered in most engineering and atmospheric configurations. In this respect, consequences of Eqs. (26) and (27) are far reaching: (i) Michael's algorithm (Sec. 2) provides a particularly efficient way of generating large k-samples of the distributions f^{ss} and f^{gs}; and (ii) for studies that require gas models other than Malkmus' model, inverse Gaussian properties of the best fitting Malkmus' model may be useful for sampling procedure design.

4. ANALYTICAL INTEGRATION FOR STRATIFIED CONFIGURATIONS

The preceding section concerns studies in which a k-distribution formulation has been chosen from the start. We now illustrate that, within studies in which this initial choice has not been made (studies that stick to the spectral narrow band average formulation), a transformation to the k-domain (which corresponds practically to an inverse Laplace transform) often facilitates the derivations because of inverse Gaussian properties.

When computing infra-red cooling rates in horizontally stratified planetary atmospheres, integrals are encountered of the following forms:[12]

$$A(l) = \int_0^L \bar{\tau}(l+z)\frac{\partial \bar{B}}{\partial z}(z)\mathrm{d}z \tag{31}$$

and

$$C(l) = \int_0^L \frac{\partial \bar{\tau}}{\partial l}(l+z)\frac{\partial \bar{B}}{\partial z}(z)\mathrm{d}z, \tag{32}$$

where z is the vertical coordinate and $\bar{B}(z)$ is the spectral average black-body intensity at altitude z. These integrals appear when considering the influence of an atmospheric layer of thickness L on the radiative balance or the radiative flux at a given altitude. The assumption is commonly made that the blackbody intensity is a polynomial function of altitude within each layer, which means that its derivative may be written as

$$\frac{\partial \bar{B}}{\partial z}(z) = \sum_{n=0}^{N} \alpha_n z^n \tag{33}$$

and Eqs. (31) and (32) are computed with standard methods of quadrature. A transformation to the k-domain allows to solve these integrals analytically whatever the polynomial order and to avoid numerical quadratures. The average transmission function and its derivative are written in terms of k-distributions:

$$\bar{\tau}(l) = \int_0^\infty \exp(-kl) f(k)\,\mathrm{d}k \tag{34}$$

and

$$\frac{\partial \bar{\tau}}{\partial l}(l) = \int_0^\infty -k\exp(-kl) f(k)\,\mathrm{d}k. \tag{35}$$

Substituting Eqs. (33)–(35) into Eqs. (31) and (32) and inverting the z and k integrals gives

$$A(l) = \sum_{n=0}^{N} \alpha_n A_n(l) \tag{36}$$

and

$$C(l) = \sum_{n=0}^{N} \alpha_n C_n(l) \tag{37}$$

with

$$A_n(l) = \int_0^\infty \exp(-kl) h_n(k) f(k)\,\mathrm{d}k \tag{38}$$

and

$$C_n(l) = \int_0^\infty -k\exp(-kl) h_n(k) f(k)\,\mathrm{d}k, \tag{39}$$

where

$$h_n(k) = \int_0^L \exp(-kz) z^n\,\mathrm{d}z. \tag{40}$$

Equation (40) leads to the following recurrence formula:

$$h_0(k) = \frac{1}{k}[1 - \exp(-kL)] \tag{41}$$

and

$$h_n(k) = -\frac{L^n}{k}\exp(-kL) + \frac{n}{k}h_{n-1}(k). \tag{42}$$

Using this recurrence formula, A and C can be expressed as weighted sums of integrals of the general form

$$I_n(d) = \int_0^\infty \frac{1}{k^n}\exp(-kd)f(k)\,\mathrm{d}k \tag{43}$$

with $d = l$ or $d = l + L$.

The remaining integral involves the k-distribution f multiplied by an exponential extinction. It has already been pointed out that such a product leads to an inverse Gaussian distribution and, with the preceding notations, $I_n(d)$ can be written as

$$I_n(d) = \bar{\tau}(d)\int_0^\infty k^{-n}f^{\mathrm{ss}}(k;d)\,\mathrm{d}k \tag{44}$$

which is the average transmission function time of the nth negative moment of an inverse Gaussian distribution of mean $\mu^*(d)$ and shape parameter $\phi^*(d)$:

$$I_n(d) = \bar{\tau}(d)\mathscr{E}_{-n}(\mu^*, \phi^*) = \bar{\tau}(d)\mu^{*-n}\sum_{s=0}^{n}\frac{(n+s)!}{(n-s)!}(2\phi^*)^{-s}. \tag{45}$$

This result provides analytical expressions for the vertical integrals A and C for all configurations in which the assumption of polynomial $\bar{B}$ profiles is meaningful. The developed expressions are given in Appendix B for linear and quadratic $\bar{B}$ profiles.

Acknowledgements—We are grateful to Pr. G. Letac and Dr. J. L. Dunau for their support and to L. Fairhead for English editing. This work was sponsored by PIRSEM/CNRS, ADEME "Service Habitat et Tertiaire" and "Minitère de la Recherche et de la Technologie."

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APPENDIX A: CUMULATIVE OF THE INVERSE GAUSSIAN RECIPROCAL

The following complex function is considered where x and b are positive parameters:

$$C(t) = \int_0^x \mathrm{e}^{\mathrm{i}tu}\mathscr{W}(u;b)\,\mathrm{d}u \tag{A1}$$

which can be expressed as (Eqs. (4) and (8))

$$C(t) = \exp\{b[1-(1-2it/b)^{+1/2}]\}\int_0^x \mathfrak{I}(u;(1-2it/b)^{-1/2}, b(1-2it/b)^{+1/2})\,\mathrm{d}u \tag{A2}$$

and Eq. (9) leads to

$$C(t) = \exp\{b[1-(1-2it/b)^{+1/2}]\}\left\{\Gamma\left[-\sqrt{\frac{b}{x}}(1-x(1-2it/b)^{+1/2})\right]\right.$$
$$\left.+\exp[2b(1-2it/b)^{+1/2}]\Gamma\left[-\sqrt{\frac{b}{x}}(1+x(1-2it/b)^{+1/2})\right]\right\}. \tag{A3}$$

Taking the first derivative of $C(t)$ and evaluating it at $t=0$, we get

$$\int_0^x u\mathscr{W}(u;b)\,\mathrm{d}u = \Gamma\left[-\sqrt{\frac{b}{x}}(1-x)\right] - \mathrm{e}^{2b}\Gamma\left[-\sqrt{\frac{b}{x}}(1+x)\right]. \tag{A4}$$

This results allows to express the cumulative distribution function of the reciprocal of the inverse Gaussian in a simple way. Indeed Eqs. (7) and (11) give

$$\int_0^y \mathscr{R}(u;a,b)\,\mathrm{d}u = \int_0^{ay} u/a\mathfrak{I}(u/a;1/a,b)\,\mathrm{d}u = \int_0^{ay} u\mathscr{W}(u;b)\,\mathrm{d}u \tag{A5}$$

and combining Eqs. (A4) and (A5) leads to Eq. (13). Equation (A4) can also be integrated by parts to get the following expression:

$$\int_0^x u^2\mathscr{W}(u;b)\,\mathrm{d}u = 1/b\int_0^x u\mathscr{W}(u;b)\,\mathrm{d}u + \int_0^x \mathscr{W}(u;b)\,\mathrm{d}u - 2/bx^2\mathscr{W}(x;b). \tag{A6}$$

This allows the derivation of the cumulative distribution of f^{gg}. First, f^{gg} is expressed in terms of the standard Wald's distribution (Eqs. (7), (8) and (28)):

$$f^{\mathrm{gg}}(k;l) = \mu^*(l)^{-3}\left(1+\frac{1}{\phi^*(l)}\right)^{-1} k^2\mathscr{W}(k/\mu^*(l);\phi^*(l)). \tag{A7}$$

Then a simple change of variable:

$$g^{\mathrm{gg}}(k;l) = \left(1+\frac{1}{\phi^*(l)}\right)^{-1}\int_0^{k/\mu^*(l)} u^2\mathscr{W}(u;\phi^*(l))\,\mathrm{d}u \tag{A8}$$

and combination of Eqs. (A6) and (A8) give

$$g^{\mathrm{gg}}(k;l) = \left(1+\frac{1}{\phi^*(l)}\right)^{-1}\left\{1/\phi^*(l)\int_0^{k/\mu^*(l)} u\mathscr{W}(u;\phi^*(l))\,\mathrm{d}u\right.$$
$$\left.+\int_0^{k/\mu^*(l)}\mathscr{W}(u;\phi^*(l))\,\mathrm{d}u - 2/\phi^*(l)\left(\frac{k}{\mu^*(l)}\right)^2\mathscr{W}(k/\mu^*(l);\phi^*(l))\right\} \tag{A9}$$

which may also be written as

$$g^{\mathrm{gg}}(k;l) = \frac{1}{1+\phi^*(l)}\left[g^{\mathrm{gs}}(k;l)+\phi^*(l)g^{\mathrm{ss}}(k;l) - 2\mu^*(l)\left(1+\frac{1}{\phi^*(l)}\right)f^{\mathrm{gg}}(k;l)\right]. \tag{A10}$$

APPENDIX B: DEVELOPED EXPRESSIONS FOR A_0, C_0, A_1 AND C_1

The following expressions are obtained on the basis of Eqs. (36)–(39) and (45):

$$A_0 = +\bar{\tau}(l)\frac{1}{\mu^*(l)}\left[1+\frac{1}{\phi^*(l)}\right] - \bar{\tau}(l+L)\frac{1}{\mu^*(l+L)}\left[1+\frac{1}{\phi^*(l+L)}\right], \tag{B1}$$

$$C_0 = -\bar{\tau}(l)+\bar{\tau}(l+L), \tag{B2}$$

$$A_1 = +\bar{\tau}(l)\frac{1}{\mu^{*2}(l)}\left[1+\frac{3}{\phi^*(l)}+\frac{6}{\phi^{*2}(l)}\right] - \bar{\tau}(l+L)\left\{\frac{1}{\mu^{*2}(l+L)}\left[1+\frac{3}{\phi^*(l+L)}+\frac{6}{\phi^{*2}(l+L)}\right]\right.$$
$$\left.+L\frac{1}{\mu^*(l+L)}\left[1+\frac{1}{\phi^*(l+L)}\right]\right\}, \tag{B3}$$

$$C_1 = -\bar{\tau}(l)\frac{1}{\mu^*(l)}\left[1+\frac{1}{\phi^*(l)}\right]+\bar{\tau}(l+L)\left\{\frac{1}{\mu^*(l+L)}\left[1+\frac{1}{\phi^*(l+L)}\right]+L\right\}, \tag{B4}$$

with

$$\mu^*(l) = \mu(1+2\mu l/\phi)^{-1/2}, \tag{B5}$$

$$\phi^*(l) = \phi(1+2\mu l/\phi)^{1/2}, \tag{B6}$$

and

$$\bar{\tau}(l) = \exp[\phi - \phi^*(l)]. \tag{B7}$$