Positive feedback in climate: stabilization or runaway, illustrated by a simple experiment.

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Supplementary material

APPENDIX A

Mathematical model of the experiment

A simple model of the device is now presented to better understand and describe the behavior of the experiment.

Each cylindrical water container is initially of mass m_2. The "horizontal bar moment" is M^H, and the "return moment" is M^R. Other definitions are given in the text.

At equilibrium, the sum of the moments of force is null: $M^H + M^R = 0$. Initially, as the system is vertically straight, the moments are nulls. Given the definition of the moment of force, if the device is tilted by $\Delta\theta$ compared to this initial state, the corresponding change in the vertical bar moment is $\Delta M^R = -g_{grav}\ l_3 m_3\ sin(\Delta\theta)$, where g_{grav} is the gravitational acceleration (absolute value). Considering that $\Delta\theta$ is relatively small, this simplifies to $\Delta M^R \approx -g_{grav}\ l_3 m_3\ \Delta\theta$. Therefore, at equilibrium, the tilt change $\Delta\theta$ in response to any change ΔM^H of the horizontal bar moment reads:

$$\Delta\theta = \frac{1}{g_{grav}\ l_3 m_3}\,\Delta M^H \qquad \text{(A1)}$$

a. No-feedback response to an external perturbation

Adding a mass m_1 to one extremity of the horizontal bar makes the device tilt by $\Delta\theta_i$ (i for initial) (Fig. 4b) and adds the moments $\Delta M_i^H = g_{grav}\ l_1 m_1\ cos(\Delta\theta_i)$, which simplifies to $\Delta M_i^H = g_{grav}\ l_1 m_1$ when considering $\Delta\theta$ small:

$$\Delta\theta_i = \frac{1}{g_{grav}\ l_3 m_3}\,\Delta M_i^H \qquad \text{(A2)}$$

1) RESPONSE WITH FEEDBACK

The next step consists in opening the tap between the two containers, allowing the positive feedback process between tilt and moment (Fig. 4c). The total variation ΔM^H of the horizontal bar moment is given by:

$$\Delta M^H = \Delta M_i^H + \Delta M_f^H \qquad \text{(A3)}$$

where ΔM_H^i is the initial perturbation and $\Delta M_f^H = g_{grav}\ l_2 \Delta m$ (subscript f for feedback) is the amplification due the mass Δm of water which has passed from the upper container to the lower one (with $\Delta\theta$ small so that $cos(\Delta\theta) \approx 1$). This mass variation is given by the difference $Z = l_2 sin(\Delta\theta) \approx l_2\Delta\theta$ between the height of the two containers:

$$\Delta m = \rho S\,\frac{Z}{2} \approx \rho S\,\frac{l_2\Delta\theta}{2}, \qquad \text{(A4)}$$

with ρ the water density, S the cross section of the containers. It follows:

$$\Delta M^H = \Delta M_i^H + g_{grav}\ \rho S\,\frac{l_2^2}{2}\,\Delta\theta \qquad \text{(A5)}$$

The variable M^H is thus a function of θ (Eq. A5) and θ is a function of M^H (Eq. A1). Combining Eqs. A1, A2 and A5, it follows:

$$\Delta\theta = \frac{1}{1-g}\Delta\theta_i = k\Delta\theta_i \qquad \text{(A6)}$$

with

$$g = \frac{\rho S\, l_2^2}{2\, l_3 m_3} \tag{A7}$$

and

$$k = \frac{1}{1-g} \tag{A8}$$

where g is the feedback gain and k is the feedback amplification factor. To further understand the meaning of the gain g, it is worth noting that Eq.A6 can also be written $\Delta\theta = \Delta\theta_i + g\Delta\theta$, with $\Delta\theta_i$ the no-feedback response and $g\Delta\theta$ the additional response due to feedbacks.

APPENDIX B

Mathematical model of the albedo feedback

The solar flux F absorbed by the surface is

$$F = I(1-\alpha), \tag{B1}$$

where I is a quarter of the solar constant and α the albedo. At equilibrium, the flux absorbed by the surface is equal to that emitted by the surface and the surface temperature is given by $T = (F/\sigma)^{1/4}$ assuming an emissivity of 1, where σ is the Stefan-Boltzmann constant. For small variations, this equation may be linearized and the temperature varies with the absorbed solar flux as follows:

$$\Delta T = -\frac{\Delta F}{\lambda_P} \tag{B2}$$

With

$$\lambda_P = -4\sigma T^3 \tag{B3}$$

As a comparison with the experiment, T is the analog of the tilt θ of the device, F is the analog of the moment of force M^H, and Eq. B2 is the analog of Eq. A1.

a. No-feedback response to an external perturbation

The absorbed solar flux is assumed to be perturbed by ΔQ (positive or negative) by some external process. In response to this perturbation, the surface temperature directly increases or decreases by ΔT_i according to Eq. B2:

$$\Delta T_i = -\frac{\Delta Q}{\lambda_P} \tag{B4}$$

This perturbation corresponds to adding the mass m_1 and keeping the tap closed in the experiment and Eq. B4 is an analog to Eq.A2.

b. Response with feedback

The resulting variation ΔF of the absorbed solar flux can be decomposed into a first part due the external perturbation ΔQ and a second part due to the albedo feedback amplification ΔF_α:

$$\Delta F = \Delta Q + \Delta F_\alpha \tag{B5}$$

ΔF_α depends on the surface albedo variation (Eq. B1) and may be expressed as a function of the surface temperature ΔT as:

$$\Delta F_\alpha = \lambda_\alpha \Delta T \tag{B6}$$

with

$$\lambda_\alpha = -I\frac{\partial \alpha}{\partial T} \tag{B7}$$

where $\lambda_\alpha > 0$ as the albedo α decreases when the surface temperature increases. Combining Eq.B2 and Eqs.B4 to B6, it follows:

$$\Delta T = \frac{1}{1-g_\alpha}\Delta T_i \tag{B8}$$

with

$$g_\alpha = -\frac{\lambda_\alpha}{\lambda_P} \tag{B9}$$

The feedback is positive as $\lambda_\alpha > 0$ and therefore $g_\alpha > 0$. Equation B8 is an analog to Eq.A6 and Eq. B9 is an analog to Eq.A7.

APPENDIX C

Experiment where the gain is doubled

Experimental device with two identical pairs of containers that communicate through two independent pipes. The two taps can be opened one after the other to observe that the tilt increase due to two feedbacks is more than two times the tilt increase due to one feedback.

Figures

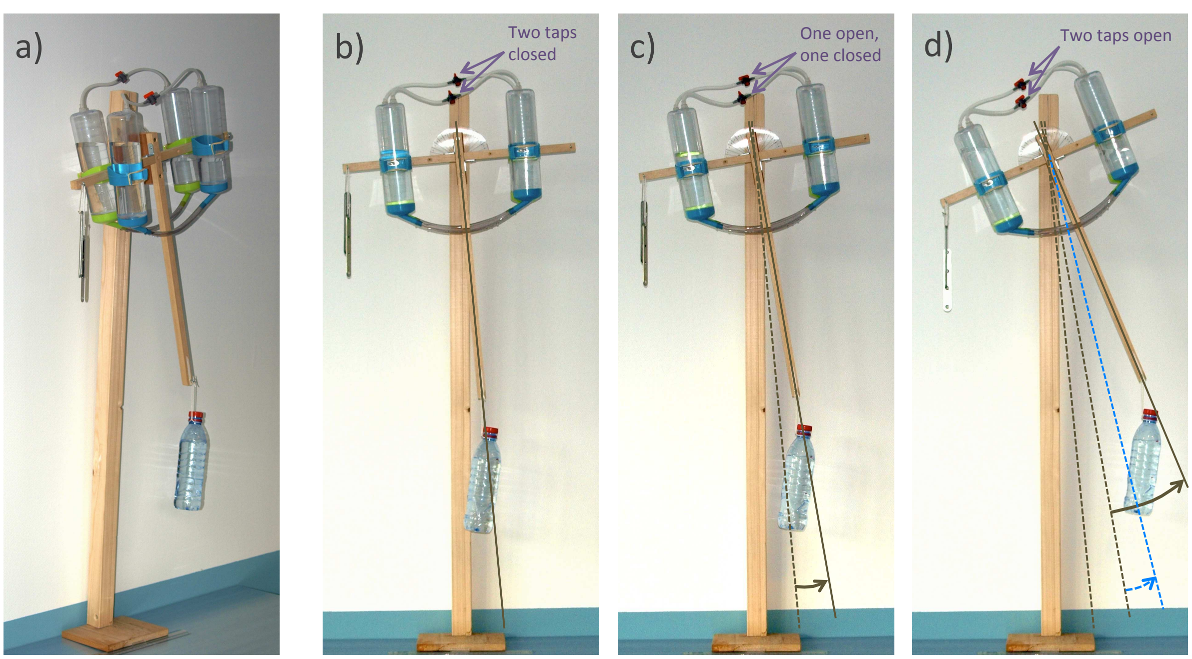

FIG. C1. **Figure C-1**: (a) General view of the device. Perturbed state with (b) the two taps closed, (c) only one tap open and (d) two taps open. On (d), three dashed lines illustrate the tilt with the two taps closed (as in (b)), the tilt when only one tap is open (as in (c)) and the tilt that would be if the additional tilt increase when opening the second tap was identical to the additional tilt increase when opening the first tap (blue line). The difference between the blue line and the actual tilt illustrate that the tilt increase due to two feedbacks is more than two times the tilt increase due to one feedback.