Differential treatment of the PBL between inner and outer regions of cold pools; II: over land

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Key Points:

- Thanks to the adaptation of surface temperature to cold pool temperature, sensible heat flux difference between inner and outer regions of cold pools is weaker than in the ocean case, which allows stronger cold pools.
- A negative feedback loop due radiation and to sensible and latent heat fluxes dampens the surface temperature adaptation.
- The difference of behaviour of cold pools over land when compared to ocean changes the land-ocean contrast of precipitation.

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Abstract

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1 Introduction

The difference between the land case and the ocean case is twofold:

1. Over land the surface temperature depends on the boundary layer temperature and humidity. Hence one may expect surface temperature and humidity to differ between the wake and the off-wake regions. Moreover these differences may feed-back on the cold pool state so that they cannot be considered as prescribed as they were in the ocean case : in the land case, the surface and moisture differences are internal variables of the system. Some new model will have to be provided to represent the processes that drive these differences.
2. Over land the soil surface may be partially saturated: this is described thanks to the aridity factor β which is the fraction of the soil surface which is saturated, the rest of the surface beeing perfectly dry. Then the surface average humidities q_s^w and q_s^x over each of the regions (w) and (x) no longer coincide with the saturation humidities $q_{\rm sat}(T_s^w)$ and $q_{\rm sat}(T_s^x)$. The link between the PBL variables, q_s^w and q_s^x, and the surface variables, β, $q_{\rm sat}(T_s^w)$ and $q_{\rm sat}(T_s^x)$, will also require some new equations.

2 The merging equations

Atmospheric boundary layer

The surface as seen by each of the two atmospheric columns is described by the enthalpy and moisture mean fluxes, ϕ and $\hat{\phi}$, and the surface temperature and humidity, T_s and q_s. Turbulent vertical diffusion equations relate mean surface fluxes with mean surface temperature and humidity; they are identical to those of the ocean case. They are composed of two sets:

First, the surface flux equations express the mean fluxes in terms of the surface variables T_s and q_s:

$$\left\{ \begin{array}{lcl} \hat{\phi^w} & = & \hat{K}^w(q_1^w \; - \; q_s^w) \\ \hat{\phi^x} & = & \hat{K}^x(q_1^x \; - \; q_s^x) \end{array} \right. \qquad \left\{ \begin{array}{lcl} \phi^w & = & C_p K^w(T_1^w \; - \; T_s^w) \\ \phi^x & = & C_p K^x(T_1^x \; - \; T_s^x) \end{array} \right. \tag{1}$$

Second, the boundary layer equations describe the link between enthalpy and humidity at first level and the surface fluxes:

$$\left\{ \begin{array}{lcl} q_1^w & = & (\hat{A^w} \; + \; \hat{B^w}\hat{\phi^w}\Delta t) \\ q_1^x & = & (\hat{A^x} \; + \; \hat{B^x}\hat{\phi^x}\Delta t) \end{array} \right. \qquad \left\{ \begin{array}{lcl} C_p T_1^w & = & (A^w \; + \; B^w\phi^w\Delta t) \\ C_p T_1^x & = & (A^x \; + \; B^x\phi^x\Delta t) \end{array} \right. \tag{2}$$

Atmosphere/soil interface

At this stage, the link between q_s^w and T_s^w is unknown (on ocean it would be $q_s^w \; = \; q_{\rm sat}(T_s^w)$). The surface evaporation is represented by the aridity coefficient β, that is the ratio of the evaporation to the potential evaporation. Consenquently $\hat{\phi^w} \; = \; \hat{K}^w\beta[q_1^w - q_{\rm sat}(T_s^w)]$.

From $\hat{\phi^w} \; = \; \hat{K}^w(q_1^w \; - \; q_s^w)$ and $\hat{\phi^w} \; = \; \hat{K}^w\beta[q_1^w - q_{\rm sat}(T_s^w)]$ one gets an expression of the mean surface humidity:

$$q_s^w \; = \; (1-\beta)q_1^w \; + \; \beta q_{\rm sat}(T_s^w)$$

In order to deal with variables meaningfull even when $\beta = 0$, the following variable will be used:

$$q_{\mathrm{sat,s}} \;=\; \beta q_{\mathrm{sat}}(T_s) \tag{3}$$

The surface flux equations become:

$$\begin{cases} \hat{\phi}^w &= \hat{K}^w(\beta q_1^w \;-\; q_{\mathrm{sat,s}}^w) \\ \hat{\phi}^x &= \hat{K}^x(\beta q_1^x \;-\; q_{\mathrm{sat,s}}^x) \end{cases} \quad \begin{cases} \phi^w &= C_p K^w(T_1^w \;-\; T_s^w) \\ \phi^x &= C_p K^x(T_1^x \;-\; T_s^x) \end{cases} \tag{4}$$

On the soil side, the interface is made of a single column. The equations relate the domain averages of the fluxes with the domain averages of the surface variables T_s^* and $q_{\mathrm{sat,s}}^*$:

$$\begin{cases} \hat{\phi}^* &= \hat{K}^a(\beta q^a - q_{\mathrm{sat,s}}^*) \\ \\ q^a &= \hat{A}^a \;+\; \hat{B}^a \hat{\phi}^* \Delta t \end{cases} \quad \begin{cases} \phi^* &= K^a(h^a - C_p T_s^*) \\ \\ h^a &= A^a \;+\; B^a \phi^* \Delta t \end{cases} \tag{5}$$

where q^a and h^a are apparent atmospheric moisture and enthalpy, $\hat{K}^a$ and K^a are apparent exchange coefficients, and $\hat{A}^a$, A^a, $\hat{B}^a$, and B^a describe the sensitivities of q^a and h^a to the mean fluxes $\hat{\phi}^*$ and ϕ^*.

The problem of determining expressions for these coefficients in terms of the coefficients within each column will be called the merging problem. The problem of determining the fluxes in each column once the domain average fluxes are known will be called the splitting problem.

Effective exchange coefficients

Eliminating variables q_1 and T_1 in equations (4) and (2) yields new surface flux equations:

$$\begin{cases} \hat{\phi}^w &= \hat{K}''^w(\beta \hat{A}^w \;-\; q_{\mathrm{sat,s}}^w) \\ \hat{\phi}^x &= \hat{K}''^x(\beta \hat{A}^x \;-\; q_{\mathrm{sat,s}}^x) \end{cases} \quad \begin{cases} \phi^w &= K'^w(A^w \;-\; C_p T_s^w) \\ \phi^x &= K'^x(A^x \;-\; C_p T_s^x) \end{cases} \tag{6}$$

where $\hat{K}''^w$, $\hat{K}''^x$, K'^w, and K'^x are the effective exchange coefficients, that is exchange coefficients accounting for the boundary layer feedbacks:

$$\begin{cases} \hat{K}''^w &= \dfrac{\hat{K}^w}{1 - \beta \hat{K}^w \hat{B}^w \Delta t} \\ \\ \hat{K}''^x &= \dfrac{\hat{K}^x}{1 - \beta \hat{K}^x \hat{B}^x \Delta t} \end{cases} \quad \begin{cases} K'^w &= \dfrac{K^w}{1 - K^w B^w \Delta t} \\ \\ K'^x &= \dfrac{K^x}{1 - K^x B^x \Delta t} \end{cases} \tag{7}$$

Expression of the domain average fluxes

Applying the second product identity to $\hat{K}''$, $(\beta \hat{A} \;-\; q_{\mathrm{sat,s}})$ and their product $\hat{\phi}$ (and similarly for ϕ) yields expressions of the domain average fluxes:

$$\begin{aligned} \hat{\phi}^* &= \hat{K}''^*(\beta \hat{A}^* - q_{\mathrm{sat,s}}^*) \;+\; \sigma_w \sigma_x \delta \hat{K}''(\beta \delta \hat{A} - \delta q_{\mathrm{sat,s}}) \\ \\ \phi^* &= K'^*(A^* - C_p T_s^*) \;+\; \sigma_w \sigma_x \delta K'(\delta A - C_p \delta T_s) \end{aligned} \tag{8}$$

In section (4) will be introduced a model of surface temperature difference. Then it will be proved in sections (4.2) and (4.3) that the moisture difference $\delta q_{\mathrm{sat,s}}$ and the temperature difference δT_s are affine functions of the surface fluxes:

$$\delta q_{\mathrm{sat,s}} \;=\; \hat{M} \;+\; \hat{N} \hat{\phi}^* \quad C_p \delta T_s \;=\; M \;+\; N \phi^* \tag{9}$$

where $M,\ N,\ \hat{M},\ \hat{N}$ are the coefficients that will be determined in sections (4.2) and (4.3). They represent the feedbacks of the surface fluxes onto the temperature difference

δT_s and on the humidity difference δq_s. Then the domain average fluxes read:

$$\begin{aligned}(1+\sigma_w\sigma_x\delta\hat{K}''\hat{N})\hat{\phi}^* &= \beta\hat{K}''^*\hat{A}^* \;+\; \sigma_w\sigma_x\delta\hat{K}''(\beta\delta\hat{A}-\hat{M}) \;-\; \hat{K}''^* q^*_{\text{sat,s}} \\ (1+\sigma_w\sigma_x\delta K'N)\phi^* &= K'^*A^* \;+\; \sigma_w\sigma_x\delta K'(\delta A - M) \;-\; K'^*C_pT^*_s\end{aligned} \tag{10}$$

Mixed boundary conditions for the surface model

Similarly to the ocean case, the boundary conditions for the surface model read:

$$\hat{\phi}^* \;=\; \hat{\mu} \;-\; \hat{\lambda}q^*_{\text{sat,s}} \qquad \phi^* \;=\; \mu \;-\; \lambda T^*_s \tag{11}$$

Comparison with equations (10) yields:

$$\begin{cases}\hat{\mu} \;=\; \beta\dfrac{\hat{K}''^*\hat{A}^* \;+\; \sigma_w\sigma_x\delta\hat{K}''(\delta\hat{A}-\frac{\hat{M}}{\beta})}{1+\sigma_w\sigma_x\delta\hat{K}''\hat{N}} \\ \hat{\lambda} \;=\; \dfrac{\hat{K}''^*}{1+\sigma_w\sigma_x\delta\hat{K}''\hat{N}}\end{cases} \qquad \begin{cases}\mu \;=\; \dfrac{K'^*A^* \;+\; \sigma_w\sigma_x\delta K'(\delta A - M)}{1+\sigma_w\sigma_x\delta K'N} \\ \lambda \;=\; \dfrac{C_pK'^*}{1+\sigma_w\sigma_x\delta K'N}\end{cases} \tag{12}$$

Mixed boundary conditions in terms of $\hat{A}^a$, $\hat{B}^a$, A^a, and B^a

Eliminating q^a and h^a in equations (5) yields:

$$\hat{\phi}^* \;=\; \frac{\hat{K}^a}{1-\beta\hat{K}^a\hat{B}^a\Delta t}(\beta\hat{A}^a \;-\; q^*_{\text{sat,s}}) \qquad \phi^* \;=\; \frac{K^a}{1-K^aB^a\Delta t}(A^a \;-\; C_pT^*_s) \tag{13}$$

Comparison of these equations with equations (11) yields:

$$\begin{cases}\hat{\mu} \;=\; \dfrac{\beta\hat{K}^a\hat{A}^a}{1-\beta\hat{K}^a\hat{B}^a\Delta t} \\ \hat{\lambda} \;=\; \dfrac{\hat{K}^a}{1-\beta\hat{K}^a\hat{B}^a\Delta t}\end{cases} \qquad \begin{cases}\mu \;=\; \dfrac{K^aA^a}{1-K^aB^a\Delta t} \\ \lambda \;=\; \dfrac{C_pK^a}{1-K^aB^a\Delta t}\end{cases} \tag{14}$$

from which one gets the expressions of $\hat{A}^a$, A^a, $\hat{B}^a$, and B^a in terms of $\hat{\mu}$, μ, $\hat{\lambda}$, and λ:

$$\begin{cases}\hat{A}^a \;=\; \dfrac{1}{\beta}\dfrac{\hat{\mu}}{\hat{\lambda}} \\ \hat{B}^a \;=\; \dfrac{1}{\beta\Delta t}[\dfrac{1}{\hat{K}^a} \;-\; \dfrac{1}{\hat{\lambda}}]\end{cases} \qquad \begin{cases}A^a \;=\; \dfrac{\mu}{\lambda} \\ B^a \;=\; \dfrac{1}{\Delta t}[\dfrac{1}{K^a} \;-\; \dfrac{C_p}{\lambda}]\end{cases} \tag{15}$$

General formulas for $\hat{A}^a$, $\hat{B}^a$, A^a, and B^a

Substituting in the last equation the expressions of $\hat{\mu}$, μ, $\hat{\lambda}$, and λ given by equations(12) yields expressions for $\hat{A}^a$, $\hat{B}^a$, A^a, and B^a in terms of $\hat{K}^a$ and K^a:

$$\begin{cases}\hat{A}^a \;=\; \hat{A}^* \;+\; \sigma_w\sigma_x\dfrac{\delta\hat{K}''}{\hat{K}''^*}(\delta\hat{A}-\dfrac{\hat{M}}{\beta}) \\ \hat{B}^a \;=\; \dfrac{1}{\beta\Delta t}[\dfrac{1}{\hat{K}^a} \;-\; \dfrac{1+\sigma_w\sigma_x\delta\hat{K}''\hat{N}}{\hat{K}''^*}]\end{cases} \qquad \begin{cases}A^a \;=\; A^* \;+\; \sigma_w\sigma_x\dfrac{\delta K'}{K'^*}(\delta A - M) \\ B^a \;=\; \dfrac{1}{\Delta t}[\dfrac{1}{K^a} \;-\; \dfrac{1+\sigma_w\sigma_x\delta K'N}{K'^*}]\end{cases} \tag{16}$$

Demanding continuity for q^a towards q_1 when $\sigma_w\sigma_x \longrightarrow 0$ yields:

$$\hat{K}^a \;=\; \hat{K}^* \qquad K^a \;=\; K^*$$

hence the general formulas for $\hat{A}^a$, $\hat{B}^a$, A^a, and B^a:

$$\begin{cases} \hat{A}^a = \hat{A}^{a0} - \dfrac{\sigma_w \sigma_x}{\beta} \dfrac{\delta \hat{K}"}{\hat{K}"^*} \hat{M} \\ \hat{B}^a = \hat{B}^{a0} - \dfrac{\sigma_w \sigma_x}{\beta} \dfrac{\delta \hat{K}"}{\hat{K}"^*} \hat{N} \dfrac{1}{\Delta t} \end{cases} \quad \begin{cases} A^a = A^{a0} - \sigma_w \sigma_x \dfrac{\delta K'}{K'^*} M \\ B^a = B^{a0} - \sigma_w \sigma_x \dfrac{\delta K'}{K'^*} N \dfrac{1}{\Delta t} \end{cases} \tag{17}$$

where:

$$\begin{cases} \hat{A}^{a0} = \hat{A}^* + \sigma_w \sigma_x \dfrac{\delta \hat{K}"}{\hat{K}"^*} \delta \hat{A} \\ \hat{B}^{a0} = \dfrac{\sigma_w \hat{K}^w \hat{K}"^w \hat{B}^w \; + \; \sigma_x \hat{K}^x \hat{K}"^x \hat{B}^x}{\hat{K}^* \hat{K}"^*} \end{cases} \quad \begin{cases} A^{a0} = A^* + \sigma_w \sigma_x \dfrac{\delta K'}{K'^*} \delta A \\ B^{a0} = \dfrac{\sigma_w K^w K'^w B^w \; + \; \sigma_x K^x K'^x B^x}{K^* K'^*} \end{cases} \tag{18}$$

where the $\hat{B}^{a0}$ and B^{a0} expressions have been determined by the same argument as in the ocean paper.

3 The Splitting equations

From the values of the variables A^a, B^a and K^a (and similarly for moisture) the surface model determines the domain average of the surface heat and evaporation fluxes. Then, boundary conditions for the atmosphere boundary layer model require surface fluxes within each of the (w) and (x) regions (Equations 2).

The third product identity applied to: (i) $\hat{K}"$, $(\beta \hat{A} - q_{\mathrm{sat,s}})$ and their product $\hat{\phi}$, (ii) K', $(A - C_p T_s)$ and their product ϕ, yields:

$$\begin{cases} \hat{K}"^* \delta \hat{\phi} - \delta \hat{K}" \hat{\phi}^* = \hat{K}"^x \hat{K}"^w (\beta \delta \hat{A} - \delta q_{\mathrm{sat,s}}) \\ K'^* \delta \phi - \delta K' \phi^* = K'^x K'^w (\delta A - C_p \delta T_s) \end{cases} \tag{19}$$

Whence the expressions of the flux differences $\delta \hat{\phi}$ and $\delta \phi$:

$$\begin{cases} \delta \hat{\phi} = \dfrac{\delta \hat{K}"}{\hat{K}"^*} \hat{\phi}^* + \dfrac{\hat{K}"^x \hat{K}"^w}{\hat{K}"^*} (\beta \delta \hat{A} - \delta q_{\mathrm{sat,s}}) \\ \delta \phi = \dfrac{\delta K'}{K'^*} \phi^* + \dfrac{K'^x K'^w}{K'^*} (\delta A - C_p \delta T_s) \end{cases} \tag{20}$$

Then the surface fluxes in the (w) and the (x) regions read:

$$\begin{cases} \hat{\phi}^w = \hat{\phi}^* + \sigma_x \delta \hat{\phi} \\ \hat{\phi}^x = \hat{\phi}^* - \sigma_w \delta \hat{\phi} \end{cases} \quad \begin{cases} \phi^w = \phi^* + \sigma_x \delta \phi \\ \phi^x = \phi^* - \sigma_w \delta \phi \end{cases} \tag{21}$$

4 Coupling with the soil model

4.1 Model of surface temperature difference

In this section we build a simple model of the surface temperature difference between the (w) and (x) regions. The model is based on the relation imposed by soil inertia between the amplitudes of heat flux and temperature variations.

The idea is that the movements of cold pools or of their gust fronts at the land surface induces a variation of the soil surface heat flux $\delta \phi_g$ during a characteristic time τ, which induces a variation δT_s of the surface temperature given by:

$$\delta T_s = \frac{\sqrt{\tau}}{I} \delta \phi_g \tag{22}$$

where I is the soil thermal inertia. The time τ is estimated as the time it takes to travel a distance equal to the radius of the cold pools at the spreading speed C_* of the pools. Since $\sigma_w = \pi r^2 D_w$, τ reads:

$$\tau = \frac{1}{C_*}\sqrt{\frac{\sigma_w}{\pi D_w}}$$

$\delta\phi_g$ decomposition: The flux ϕ_g is related with the sensible heat flux ϕ, the evaporation flux $\hat{\phi}$, and the net radiation flux R_n by:

$$\phi_g = \phi + L_v\hat{\phi} + R_n$$

hence its difference $\delta\phi$ between regions (w) and (x) reads:

$$\delta\phi_g = \delta\phi + L_v\hat{\delta\phi} + \delta R_n \tag{23}$$

Temperature difference δT_s expression: The substitution of (22) in (23) yields the surface temperature difference equation:

$$\delta T_s = \frac{\sqrt{\tau}}{I}[\delta\phi + L_v\hat{\delta\phi} + \delta R_n] \tag{24}$$

4.2 Enthalpy coupling equations

The purpose of this section is to express each of the flux differences $\delta\phi$, $\hat{\delta\phi}$, and δR_n as linear combinations of δT_s and ϕ^*. Non-linearities will be (partly) accounted for by using two distinct linearizations of $q_{\text{sat}}(T_s)$ and $R_{\text{Lu}}(T_s)$ in the vicinity of $T_{\text{s}}^{0,w}$ and $T_{\text{s}}^{0,x}$.

In the following we shall write equations relating average values of fields over the (w) or the (x) region, these relations being valid equally over the two regions. For the sake of simplicity, the relations wiil be written only once, with average values of the fields written with a '+' character as a superscript in lieu of the 'w' or 'x' superscript.

δR_n *expression*

The net radiation R_n at the surface is the sum of the net short wave radiation R_{Sn} and of the net long wave radiation R_{Ln}:

$$R_n = R_{\text{Sn}} + R_{\text{Ln}}$$

hence:

$$\delta R_n = \delta R_{\text{Sn}} + \delta R_{\text{Ln}}$$

We assume that $\delta R_{\text{Sn}} \simeq 0$. Then δR_n reduces to δR_{Ln}.

The net long wave radiative flux is the difference between the downwelling long wave radiation R_{Ld} and the flux emitted by the surface:

$$R_{\text{Ln}}^+ = R_{\text{Ld}}^+ - \sigma(T_s^+)^4$$

We approximate R_{Ld}^+ by the radiation field from a grey body, with emissivity ϵ_1, at the temperature T_1^+ of the first model layer:

$$R_{\text{Ld}}^+ = \epsilon_1\sigma(T_1^+)^4$$

T_1^+ may be expressed in terms of T_s^+ by combining equations (1) and (2):

$$\left\{\begin{array}{l} \phi^+ = C_p K^w(T_1^+ - T_s^+) \\ \\ C_p T_1^+ = (A^+ + B^+\phi^+\Delta t) \end{array}\right\} \implies T_1^+ = \frac{A^+ - C_p K B^+ \Delta t T_s^+}{C_p(1 - K^+ B^+ \Delta t)} \tag{25}$$

Definition of the reference state

We define the reference state by the initial surface temperatures $T_{\rm s}^{0,+}$ from which one may define:

$$q_{\rm sat,s}^{0,+} \;=\; \beta q_{\rm sat}(T_{\rm s}^{0,+})$$

Then the reference fluxes $\phi^{0,+}$, $\hat{\phi}^{0,+}$, R_n^{0+} read:

$$\begin{aligned}\phi^{0,+} &= K'^{+}(A^{+} - C_p T_{\rm s}^{0,+})\\ \hat{\phi}^{0,+} &= \hat{K}"^{+}(\beta \hat{A}^{+} - q_{\rm sat,s}^{0,+})\\ R_{\rm Ln}^{0,+} &= R_1^{0+} - R_s^{0+}\end{aligned} \tag{26}$$

where:

$$\begin{aligned}R_1^{0+} &= \epsilon_1 \sigma (T_1^{0,+})^4\\ R_s^{0+} &= \sigma (T_{\rm s}^{0,+})^4\end{aligned} \tag{27}$$

and where the reference temperature $T_1^{0,+}$ reads:

$$T_1^{0,+} \;=\; \frac{A^{+} - C_p K^{+} B^{+} \Delta t T_{\rm s}^{0,+}}{C_p(1 - K^{+} B^{+} \Delta t)} \tag{28}$$

Linearization

The saturation humidity is linearized following the formulas:

$$q_{\rm sat,s}^{+} \;=\; q_{\rm sat,s}^{0,+} \;+\; \beta \partial_T q_{\rm sat}^{+} (T_s^{+} - T_{\rm s}^{0,+}) \tag{29}$$

where

$$\partial_T q_{\rm sat}^{+} \;=\; \partial_T q_{\rm sat}(T_{\rm s}^{0,+}) \tag{30}$$

The radiation fluxes from the surface are linearized thanks to:

$$\sigma (T_s^{+})^4 \;=\; \sigma (T_{\rm s}^{0,+})^4 \;+\; R_s'^{+} (T_s^{+} - T_{\rm s}^{0,+}) \tag{31}$$

with:

$$R_s'^{+} \;=\; 4\sigma (T_{\rm s}^{0,+})^3 \tag{32}$$

The radiation fluxes from the first level of the atmosphere are linearized thanks to:

$$\epsilon_1 \sigma (T_1^{+})^4 \;=\; \epsilon_1 \sigma (T_1^{0,+})^4 \;+\; R_1'^{+} (T_1^{+} - T_1^{0,+}) \tag{33}$$

with:

$$R_1'^{+} \;=\; 4\epsilon_1 \sigma (T_1^{0,+})^3 \tag{34}$$

Expressions of the fluxes

The expression of ϕ^{+} in terms of the reference flux $\phi^{0,+}$ and the reference temperature $T_{\rm s}^{0,+}$ comes directly from:

$$\begin{aligned}\phi^{+} &= K'^{+}(A^{+} - C_p T_s^{+})\\ \phi^{0,+} &= K'^{+}(A^{+} - C_p T_{\rm s}^{0,+})\end{aligned} \tag{35}$$

yielding:

$$\phi^{+} \;=\; \phi^{0,+} - C_p K'^{+} (T_s^{+} - T_{\rm s}^{0,+}) \tag{36}$$

The expression of $\hat{\phi}^{+}$ in terms of the reference flux $\hat{\phi}^{0,+}$ and the reference temperature $T_{\rm s}^{0,+}$ comes from:

$$\begin{aligned}\hat{\phi}^{+} &= \hat{K}"^{+}(\beta \hat{A}^{+} - q_{\rm sat,s}^{+})\\ \hat{\phi}^{0,+} &= \hat{K}"^{+}(\beta \hat{A}^{+} - q_{\rm sat,s}^{0,+})\end{aligned} \tag{37}$$

combined with the linearization equation (29), yielding:

$$\hat{\phi}^{+} \;=\; \hat{\phi}^{0,+} - \beta \hat{K}^{"+} \partial_T q_{\mathrm{sat}}^{+} (T_s^{+} - T_{\mathrm{s}}^{0,+}) \tag{38}$$

The expression of R_{Ln}^{+} in terms of the refernce flux $R_{\mathrm{Ln}}^{0,+}$ and the reference temperature $T_{\mathrm{s}}^{0,+}$ comes from:

$$\begin{aligned} R_{\mathrm{Ln}}^{+} &= R_1^{+} - R_s^{+} \\ R_1^{+} &= R_1^{0+} + R_1^{'+}(T_1^{+} - T_1^{0,+}) \\ R_s^{+} &= R_s^{0+} + R_s^{'+}(T_s^{+} - T_{\mathrm{s}}^{0,+}) \end{aligned} \tag{39}$$

combined with the expression of $T_1^{0,+}$ given by equation (28) and the expression for T_1^{+} given by equation (25), yielding:

$$R_n^{+} \;=\; R_1^{0+} - R_s^{0+} - (R_1^{'+} K^{'+} B^{+} \Delta t + R_s^{'+})(T_s^{+} - T_{\mathrm{s}}^{0,+}) \tag{40}$$

General flux expressions

After linearization, each of the fluxes ϕ^{+}, $\hat{\phi}^{+}$ and R_n^{+} may be expressed in the general form:

$$\psi^{+} \;=\; \psi^{0,+} \;-\; H_{\psi}^{+}(T_s^{+} - T_{\mathrm{s}}^{0,+})$$

where ψ stands of any of the fluxes ϕ, $\hat{\phi}$ and R_n and H_{ψ}^{+} stands for one of:

- $H_{\phi}^{+} \;=\; C_p K^{'+}$
- $H_{\hat{\phi}}^{+} \;=\; \beta \hat{K}^{"+} \partial_T q_{\mathrm{sat}}^{+}$
- $H_{R_n}^{+} \;=\; R_1^{'+} K^{'+} B^{+} \Delta t \;+\; R_s^{'+}$

Then, the first product identity yields for each of the three fluxes:

$$\delta\psi \;=\; \delta\psi^0 \;-\; [H_{\psi}^{*} + (\sigma_x - \sigma_w)\delta H_{\psi}](\delta T_s - \delta T_{\mathrm{s}}^0) \;-\; \delta H_{\psi}(T_s^{*} - T_{\mathrm{s}}^{0,*})$$

and the second product identity yields for the sensible flux:

$$\phi^{*} \;=\; \phi^{0,*} \;-\; \sigma_w \sigma_x \delta H_{\phi}(\delta T_s - \delta T_{\mathrm{s}}^0) \;-\; H_{\phi}^{*}(T_s^{*} - T_{\mathrm{s}}^{0,*})$$

Thanks to this last equation it is possible to express $T_s^{*} - T_{\mathrm{s}}^{0,*}$ in terms of ϕ^{*} and $\delta T_s - \delta T_{\mathrm{s}}^0$:

$$T_s^{*} - T_{\mathrm{s}}^{0,*} \;=\; \frac{-1}{H_{\phi}^{*}}[\phi^{*} \;-\; \phi^{0,*} \;+\; \sigma_w \sigma_x \delta H_{\phi}(\delta T_s - \delta T_{\mathrm{s}}^0)]$$

Substituting this expression in the $\delta\psi$ equation yields the sought for expressions of $\delta\phi$, $\delta\hat{\phi}$ and δR_n in terms of δT_s and ϕ^{*}:

$$\delta\psi \;=\; \delta\psi^0 \;-\; [H_{\psi}^{*} + (\sigma_x - \sigma_w)\delta H_{\psi} - \sigma_w \sigma_x \frac{\delta H_{\psi} \delta H_{\phi}}{H_{\phi}^{*}}](\delta T_s - \delta T_{\mathrm{s}}^0) \;+\; \frac{\delta H_{\psi}}{H_{\phi}^{*}}(\phi^{*} - \phi^{0,*}) \tag{41}$$

4.2.1 Final formulas for the enthalpy

The substitution in equation(24) of the expressions of $\delta\phi$, $\delta\hat{\phi}$ and δR_n given by equation (41) yields:

$$\begin{aligned} \delta T_s \;=\; & \frac{\sqrt{\tau}}{I}[\delta\phi^0 \;+\; L_v \delta\hat{\phi}^0 \;+\; \delta R_n^0] \\ & -\frac{\sqrt{\tau}}{I}[H_{\phi}^{*} \;+\; L_v H_{\hat{\phi}}^{*} \;+\; H_{R_n}^{*} + (\sigma_x - \sigma_w - \sigma_w \sigma_x \frac{\delta H_{\phi}}{H_{\phi}^{*}})(\delta H_{\phi} \;+\; L_v \delta H_{\hat{\phi}} \;+\; \delta H_{R_n})](\delta T_s - \delta T_{\mathrm{s}}^0) \\ & +\frac{\sqrt{\tau}}{I H_{\phi}^{*}}[\delta H_{\phi} \;+\; L_v \delta H_{\hat{\phi}} \;+\; \delta H_{R_n}](\phi^{*} - \phi^{0,*}) \end{aligned} \tag{42}$$

Setting:

$$\begin{cases} \delta T_{\text{s,ins}} & = \dfrac{\sqrt{\tau}}{I}[\delta\phi^0 \ + \ L_v\delta\hat{\phi}^0 \ + \ \delta R_n^0] \\ \\ g & = -\dfrac{\sqrt{\tau}}{I}[H_\phi^* \ + \ L_v H_{\hat{\phi}}^* \ + \ H_{R_n}^* + (\sigma_x - \sigma_w - \sigma_w\sigma_x\dfrac{\delta H_\phi}{H_\phi^*})(\delta H_\phi \ + \ L_v\delta H_{\hat{\phi}} \ + \ \delta H_{R_n})] \\ \\ \Gamma^\phi & = +\dfrac{\sqrt{\tau}}{I H_\phi^*}[\delta H_\phi \ + \ L_v\delta H_{\hat{\phi}} \ + \ \delta H_{R_n}] \end{cases} \tag{43}$$

equation (42) becomes:

$$(1 \ - \ g)(\delta T_s \ - \ \delta T_{\text{s}}^0) \ = \ \delta T_{\text{s,ins}} \ - \ \delta T_{\text{s}}^0 \ + \ \Gamma^\phi(\phi^* - \phi^{0,*}) \tag{44}$$

4.2.2 A^a and B^a coefficients

Equation (44) is similar to equation (9) with $M \ = \ C_p[\delta T_{\text{s}}^0 + (\delta T_{\text{s,ins}} - \delta T_{\text{s}}^0 - \Gamma^\phi\phi^{0,*})/(1 - g)]$ and $N \ = \ C_p\Gamma^\phi/(1 - g)$. Then coefficients A^a and B^a are given by equation (17):

$$\begin{cases} A^a & = A^{a0} \ - \ \sigma_w\sigma_x\dfrac{C_p\delta K'}{K'^*}[\dfrac{\delta T_{\text{s,ins}} - \delta T_{\text{s}}^0 - \Gamma^\phi\phi^{0,*})}{1-g} + \delta T_{\text{s}}^0] \\ \\ B^a & = B^{a0} \ - \ \sigma_w\sigma_x\dfrac{\delta K'}{K'^*}\dfrac{C_p\Gamma^\phi}{1-g}\dfrac{1}{\Delta t} \end{cases} \tag{45}$$

4.3 Moisture coupling equation

For moisture, the boundary conditions for the surface model (11) relate $q_{\text{sat,s}}^*$ and $\hat{\phi}^*$. Hence the purpose of the present section is to translate the surface temperature difference equation (24) into an equation relating $q_{\text{sat,s}}^*$ and $\hat{\phi}^*$. First the temperature difference δT_s will be expressed in terms of the moisture difference $\delta q_{\text{sat,s}}$. Second the flux differences $\delta\phi$, $\delta\hat{\phi}$, and δR_n will be expressed as linear combinations of $\delta q_{\text{sat,s}}$ and $\hat{\phi}^*$.

Expressing δT_s in terms of $\delta q_{\text{sat,s}}$

The temperature T_s^+ may be expressed in terms of $q_{\text{sat,s}}^+$ thanks to the linearization equation (29):

$$\beta(T_s^+ \ - \ T_{\text{s}}^{0,+}) \ = \ Q^+(q_{\text{sat,s}}^+ \ - \ q_{\text{sat,s}}^{0,+})$$

where we have set:

$$Q^+ \ = \ \frac{1}{\partial_T q_{\text{sat}}^+} \tag{46}$$

The first product identity yields:

$$\beta(\delta T_s \ - \ \delta T_{\text{s}}^0) \ = \ [Q^* + (\sigma_x - \sigma_w)\delta Q](\delta q_{\text{sat,s}} - \delta q_{\text{sat,s}}^0) \ + \ \delta Q(q_{\text{sat,s}}^* - q_{\text{sat,s}}^{0,*}) \tag{47}$$

The surface temperature difference equation in terms of $\delta q_{\text{sat,s}}$

Multiplying both sides of equation (24) by β and using equation (47) yields:

$$[Q^* + (\sigma_x - \sigma_w)\delta Q](\delta q_{\text{sat,s}} - \delta q_{\text{sat,s}}^0) \ + \ \delta Q(q_{\text{sat,s}}^* - q_{\text{sat,s}}^{0,*}) \ = \ \beta\frac{\sqrt{\tau}}{I}[\delta\phi + L_v\delta\hat{\phi} + \delta R_n] \ - \ \beta\delta T_{\text{s}}^0 \tag{48}$$

After some reordering and division by Q^* one gets the surface moisture difference equation:

$$\delta q_{\text{sat,s}} - \delta q_{\text{sat,s}}^0 \ = \ -(\sigma_x - \sigma_w)\frac{\delta Q}{Q^*}(\delta q_{\text{sat,s}} - \delta q_{\text{sat,s}}^0) - \frac{\beta}{Q^*}\delta T_{\text{s}}^0 + \frac{\sqrt{\tau}}{I Q^*}[\beta\delta\phi + L_v\beta\delta\hat{\phi} + \beta\delta R_n] - \frac{\delta Q}{Q^*}(q_{\text{sat,s}}^* - q_{\text{sat,s}}^{0,*}) \tag{49}$$

Similarly to the enthalpy case, the method will consist in expressing first the flux differences $\beta\delta\phi$, $\beta\hat{\delta\phi}$ and $\beta\delta R_n$ in terms of $\delta q_{\mathrm{sat,s}} - \delta q^0_{\mathrm{sat,s}}$ and $q^*_{\mathrm{sat,s}} - q^{0,*}_{\mathrm{sat,s}}$ and expressing in turn $q^*_{\mathrm{sat,s}} - q^{0,*}_{\mathrm{sat,s}}$ in terms of $\delta q_{\mathrm{sat,s}} - \delta q^0_{\mathrm{sat,s}}$ and $\hat{\phi}^* - \hat{\phi}^{0,*}$.

General flux expressions

Similarly to the enthalpy case, the linearized flux expressions read:

$$\beta\psi^+ \;=\; \beta\psi^{0,+} \;-\; \hat{H}^+_\psi(q^+_{\mathrm{sat,s}} - q^{0,+}_{\mathrm{sat,s}})$$

where ψ stands of any of the fluxes ϕ, $\hat{\phi}$ and R_n and $\hat{H}^+_\psi$ stands for one of:

- $\hat{H}^+_\phi \;=\; C_p K'^+ Q^+$
- $\hat{H}^+_{\hat{\phi}} \;=\; \beta \hat{K}"^+$
- $\hat{H}^+_{R_n} \;=\; (R'^+_1 K'^+ B^+ \Delta t \;+\; R'^+_s)Q^+$

Then, the first product identity yields for each of the three fluxes:

$$\beta\delta\psi \;=\; \beta\delta\psi^0 \;-\; [\hat{H}^*_\psi + (\sigma_x - \sigma_w)\delta\hat{H}_\psi](\delta q_{\mathrm{sat,s}} - \delta q^0_{\mathrm{sat,s}}) \;-\; \delta\hat{H}_\psi(q^*_{\mathrm{sat,s}} - q^{0,*}_{\mathrm{sat,s}})$$

and the second product identity yields for the latent flux:

$$\beta\hat{\phi}^* \;=\; \beta\hat{\phi}^{0,*} \;-\; \sigma_w\sigma_x\delta\hat{H}_{\hat{\phi}}(\delta q_{\mathrm{sat,s}} - \delta q^0_{\mathrm{sat,s}}) \;-\; \hat{H}^*_{\hat{\phi}}(q^*_{\mathrm{sat,s}} - q^{0,*}_{\mathrm{sat,s}})$$

Thanks to this last equation it is possible to express $q^*_{\mathrm{sat,s}} - q^{0,*}_{\mathrm{sat,s}}$ in terms of $\hat{\phi}^*$ and $\delta q_{\mathrm{sat,s}} - \delta q^0_{\mathrm{sat,s}}$:

$$q^*_{\mathrm{sat,s}} - q^{0,*}_{\mathrm{sat,s}} \;=\; \frac{-\beta}{\hat{H}^*_{\hat{\phi}}}[\hat{\phi}^* \;-\; \hat{\phi}^{0,*}] \;-\; \sigma_w\sigma_x\frac{\delta\hat{H}_{\hat{\phi}}}{\hat{H}^*_{\hat{\phi}}}(\delta q_{\mathrm{sat,s}} - \delta q^0_{\mathrm{sat,s}})$$

Substituting this expression in the $\delta\psi$ equation yields the sought for expressions of $\beta\delta\phi$, $\beta\hat{\delta\phi}$ and $\beta\delta R_n$ in terms of $\delta q_{\mathrm{sat,s}}$ and $\hat{\phi}^*$:

$$\beta\delta\psi \;=\; \beta\delta\psi^0 \;-\; [\hat{H}^*_\psi + (\sigma_x - \sigma_w)\delta\hat{H}_\psi - \sigma_w\sigma_x\frac{\delta\hat{H}_\psi\delta\hat{H}_{\hat{\phi}}}{\hat{H}^*_{\hat{\phi}}}](\delta q_{\mathrm{sat,s}} - \delta q^0_{\mathrm{sat,s}}) \;+\; \beta\frac{\delta\hat{H}_\psi}{\hat{H}^*_{\hat{\phi}}}(\hat{\phi}^* - \hat{\phi}^{0,*}) \tag{50}$$

4.3.1 Final formulas for moisture

The substitution in equation(49) of the expressions of $\delta\phi$, $\hat{\delta\phi}$ and δR_n given by equation (50) yields:

$$\begin{aligned}
\delta q_{\mathrm{sat,s}} - \delta q^0_{\mathrm{sat,s}} \;=\; & \frac{\beta\sqrt{\tau}}{IQ^*}[\delta\phi^0 \;+\; L_v\delta\hat{\phi}^0 \;+\; \delta R^0_n] - \frac{\beta}{Q^*}\delta T^0_{\mathrm{s}} \\
& -[(\sigma_x - \sigma_w)\frac{\delta Q}{Q^*} - \sigma_w\sigma_x\frac{\delta Q}{Q^*}\frac{\delta\hat{H}_{\hat{\phi}}}{\hat{H}^*_{\hat{\phi}}} \\
& +\frac{\sqrt{\tau}}{IQ^*}[\hat{H}^*_\phi + L_v\hat{H}^*_{\hat{\phi}} + \hat{H}^*_{R_n} + (\sigma_x - \sigma_w - \sigma_w\sigma_x\frac{\delta\hat{H}_{\hat{\phi}}}{\hat{H}^*_{\hat{\phi}}})(\delta\hat{H}_\phi + L_v\delta\hat{H}_{\hat{\phi}} + \delta\hat{H}_{R_n})]](\delta q_{\mathrm{sat,s}} - \delta q^0_{\mathrm{sat,s}}) \\
& \frac{\delta Q}{Q^*}\frac{\beta}{\hat{H}^*_{\hat{\phi}}} + [\frac{\beta\sqrt{\tau}}{IQ^*\hat{H}^*_{\hat{\phi}}}(\delta\hat{H}_\phi \;+\; L_v\delta\hat{H}_{\hat{\phi}} \;+\; \delta\hat{H}_{R_n})](\hat{\phi}^* - \hat{\phi}^{0,*})
\end{aligned} \tag{51}$$

Setting:

$$\begin{cases} \delta q_{\text{sats,ins}} &= \dfrac{\beta\sqrt{\tau}}{Q^* I}[\delta\phi^0 \ + \ L_v\delta\hat{\phi}^0 \ + \ \delta R_n^0] \\ \hat{g} &= -\{(\sigma_x - \sigma_w)\dfrac{\delta Q}{Q^*} - \sigma_w\sigma_x \dfrac{\delta Q}{Q^*}\dfrac{\delta\hat{K}"}{\hat{K}"*} \\ & +\dfrac{\sqrt{\tau}}{Q^* I}[\hat{H}^*_\phi + L_v\hat{H}^*_{\hat{\phi}} + \hat{H}^*_{R_n} + (\sigma_x - \sigma_w - \sigma_w\sigma_x\dfrac{\delta\hat{K}"}{\hat{K}"*})(\delta\hat{H}_\phi + L_v\delta\hat{H}_{\hat{\phi}} + \delta\hat{H}_{R_n})]\} \\ \Gamma^{\hat{\phi}} &= \dfrac{1}{Q^*}\dfrac{\delta Q}{\hat{K}"*} + \dfrac{1}{Q^*}\dfrac{\sqrt{\tau}}{I\hat{K}"*}(\delta\hat{H}_\phi \ + \ L_v\delta\hat{H}_{\hat{\phi}} \ + \ \delta\hat{H}_{R_n}) \end{cases} \tag{52}$$

equation (51) becomes:

$$(1 \ - \ \hat{g})(\delta q_{\text{sat,s}} \ - \ \delta q^0_{\text{sat,s}}) \ = \ \delta q_{\text{sats,ins}} \ - \frac{\beta}{Q^*}\delta T^0_{\text{s}} \ + \ \Gamma^{\hat{\phi}}(\hat{\phi}^* - \hat{\phi}^{0,*}) \tag{53}$$

where we have used:

$$\frac{\delta\hat{H}_{\hat{\phi}}}{\hat{H}^*_{\hat{\phi}}} = \frac{\delta\hat{K}"}{\hat{K}"*} \quad \text{and} \quad \frac{\beta}{\hat{H}^*_{\hat{\phi}}} = \frac{1}{\hat{K}"*}$$

.

4.3.2 $\hat{A}^a$ and $\hat{B}^a$ coefficients

Equation (53) is similar to equation (9) with $\hat{M} = \delta q^0_{\text{sat,s}} + (\delta q_{\text{sats,ins}} - \frac{\beta}{Q^*}\delta T^0_{\text{s}} - \Gamma^{\hat{\phi}}\hat{\phi}^{0,*})/(1-\hat{g})$ and $\hat{N} = \Gamma^{\hat{\phi}}/(1-\hat{g})$. Then coefficients $\hat{A}^a$ and $\hat{B}^a$ are given by equation (17):

$$\begin{cases} \hat{A}^a &= \hat{A}^{a0} \ - \ \dfrac{\sigma_w\sigma_x}{\beta}\dfrac{\delta\hat{K}"}{\hat{K}"*}[\dfrac{(\delta q_{\text{sats,ins}} - \frac{\beta}{Q^*}\delta T^0_{\text{s}} - \Gamma^{\hat{\phi}}\hat{\phi}^{0,*})}{1-\hat{g}} + \delta q^0_{\text{sat,s}}] \\ \hat{B}^a &= \hat{B}^{a0} \ - \ \dfrac{\sigma_w\sigma_x}{\beta}\dfrac{\delta\hat{K}"}{\hat{K}"*}\dfrac{\Gamma^{\hat{\phi}}}{1-\hat{g}}\dfrac{1}{\Delta t} \end{cases} \tag{54}$$

5 1D simulations

6 Conclusion

A: The three product identities

a, b and p being three fields such that $p^w \ = \ a^w b^w$ and $p^x \ = \ a^x b^x$, the three product identities read:

$$\delta p = \ a^* \, \delta b + b^* \, \delta a + (\sigma_x - \sigma_w)\delta a \, \delta b \tag{A.1}$$

$$p^* = \ a^* b^* + \sigma_w\sigma_x\delta a \, \delta b \tag{A.2}$$

$$a^* \, \delta p \ - \ p^* \, \delta a \ = \ a^x a^w \, \delta b \tag{A.3}$$

When using any of these identities for the fields a, b and p, we shall say: "applying the first (or second, or third) product identity to the fields a, b and their product p ...". This is unambiguous for the first two identities, since a and b play identical roles. For the third one we use the convention that the field appearing solely on the right hand side is the second field.