Zonally-averaged transport equation

For a whatever scalar variable Q, the transport equation (with pressure as vertical coordinate) can be written as:

$$\frac{\partial Q}{\partial t} + u\frac{\partial Q}{\partial x} + v\frac{\partial Q}{\partial y} + \omega\frac{\partial Q}{\partial p} = S$$

where ω is the vertical velocity dp/dt, S is the source (or sink) for the variable Q. This is the advection form of transport equation.

The continuity equation (multiplied by Q) gives:

$$Q\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p}\right) = 0$$

The addition of the two equations gives:

$$\frac{\partial Q}{\partial t} + \left(\frac{\partial}{\partial x}(uQ) + \frac{\partial}{\partial y}(vQ) + \frac{\partial}{\partial p}(\omega Q)\right) = S$$

The terms in parenthesis represent the divergence of flux for the variable Q. This is the flux form of the transport equation.

We can now apply the zonal average operator:

$$\frac{\partial [Q]}{\partial t} + \frac{\partial [vQ]}{\partial y} + \frac{\partial [\omega Q]}{\partial p} = [S]$$

With the relationship $[vQ] = [v][Q] + [v^\star Q^\star]$, we obtain:

$$\frac{\partial [Q]}{\partial t} + \frac{\partial}{\partial y}([v][Q]) + \frac{\partial}{\partial y}([v^\star Q^\star]) + \frac{\partial}{\partial p}([\omega][Q]) + \frac{\partial}{\partial p}([\omega^\star Q^\star]) = [S]$$

One can use the continuity equation

$$\frac{\partial [v]}{\partial y} + \frac{\partial [\omega]}{\partial p} = 0$$

to write the transport equation as:

$$\frac{\partial [Q]}{\partial t} + \left([v]\frac{\partial [Q]}{\partial y} + [\omega]\frac{\partial [Q]}{\partial p} \right) + \left(\frac{\partial}{\partial y}([v^\star Q^\star]) + \frac{\partial}{\partial p}([\omega^\star Q^\star]) \right) = [S]$$

Terms in the first parenthesis represent the advection of the mean circulation. Terms in the second parenthesis represent the divergence of eddy fluxes.

Zonally-averaged meridional circulation equation

By using two basic equations, one is dynamic and another thermodynamic, we can deduce the equation that governs the meridional circulation:

$$\frac{du}{dt} - fv + \frac{\partial \Phi}{\partial x} = F_x$$

$$\frac{dT}{dt} - \frac{RT}{C_p p}\omega = \frac{J}{C_p}$$

They can be changed to:

$$\frac{\partial [u]}{\partial t} + \left([v]\frac{\partial [u]}{\partial y} + [\omega]\frac{\partial [u]}{\partial p} \right) + \left(\frac{\partial}{\partial y}([v^\star u^\star]) + \frac{\partial}{\partial p}([\omega^\star u^\star]) \right) = f[v] + [F_x]$$

$$\frac{\partial [T]}{\partial t} + \left([v]\frac{\partial [T]}{\partial y} + [\omega]\frac{\partial [T]}{\partial p} \right) + \left(\frac{\partial}{\partial y}([v^\star T^\star]) + \frac{\partial}{\partial p}([\omega^\star T^\star]) \right) = \frac{RT}{C_p p}[\omega] + \frac{[J]}{C_p}$$

We need the following relationships (through scale analysis) to simplify furthermore the basic equations:

$$[\omega]\frac{\partial[u]}{\partial p} \ll [v]\frac{\partial[u]}{\partial y}$$

$$\frac{\partial[\omega^\star u^\star]}{\partial p} \ll \frac{\partial[v^\star u^\star]}{\partial y}$$

$$\frac{\partial[u]}{\partial y} \ll f$$

Finally, we obtain:

$$\frac{\partial[u]}{\partial t} = f[v] - \frac{\partial[v^\star u^\star]}{\partial y} + [F_x]$$

In the same manner, the thermodynamic equation can be transformed to:

$$\frac{\partial[T]}{\partial t} = \left(\frac{RT}{C_p p} - \frac{\partial[T]}{\partial p}\right)[\omega] - \frac{\partial[v^\star T^\star]}{\partial y} + \frac{[J]}{C_p}$$

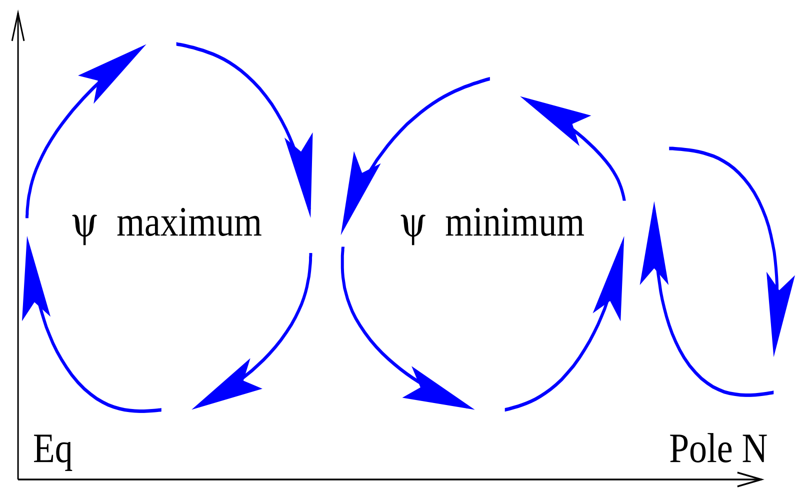
Meridional overturning circulation stream function

We can observe that the mean meridional mass circulation is non-divergent in the meriodional plane. Thus it can be represented in terms of a meridional mass transport streamfunction ψ. $[v]$ and $[\omega]$ can be calculated by:

$$[v] \equiv \frac{\partial \psi}{\partial p}; [\omega] \equiv -\frac{\partial \psi}{\partial y}$$

This satisfies the continuity equation.

$$\frac{\partial [v]}{\partial y} + \frac{\partial [\omega]}{\partial p} = 0$$



How to calculate the stream function

Mass continuity for the meridional circulation is:

$$\frac{\partial[\overline{v}]}{\partial y} + \frac{\partial[\overline{\omega}]}{\partial p} = 0$$

where ω is the vertical wind dp/dt.

We can now introduce the stream function ψ to have the following equations:

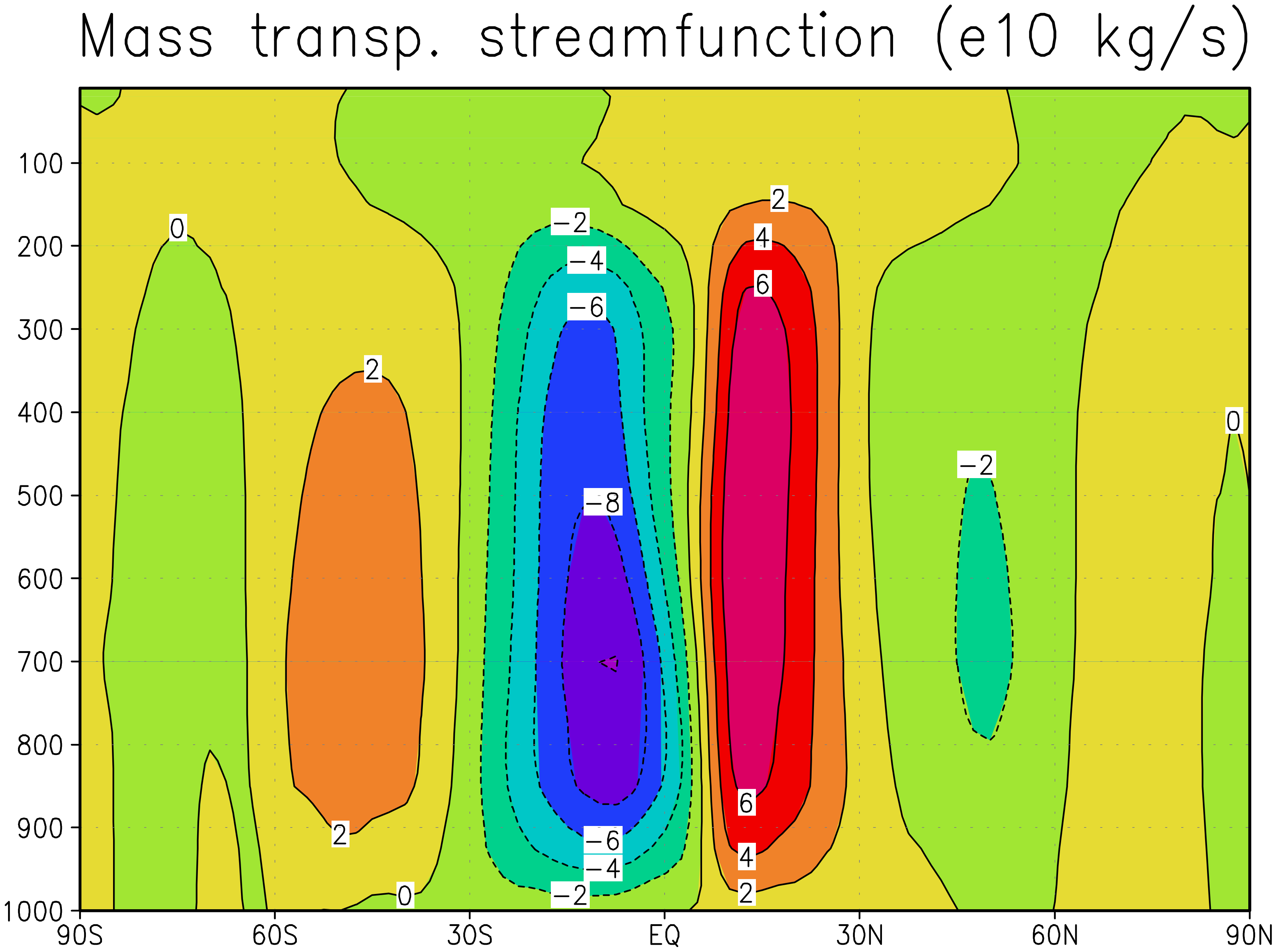
$$[\overline{v}] = \frac{g}{2\pi a \cos\phi} \frac{\partial[\psi]}{\partial p}$$

and

$$[\overline{\omega}] = \frac{-g}{2\pi a^2 \cos\phi} \frac{\partial[\psi]}{\partial \phi}$$

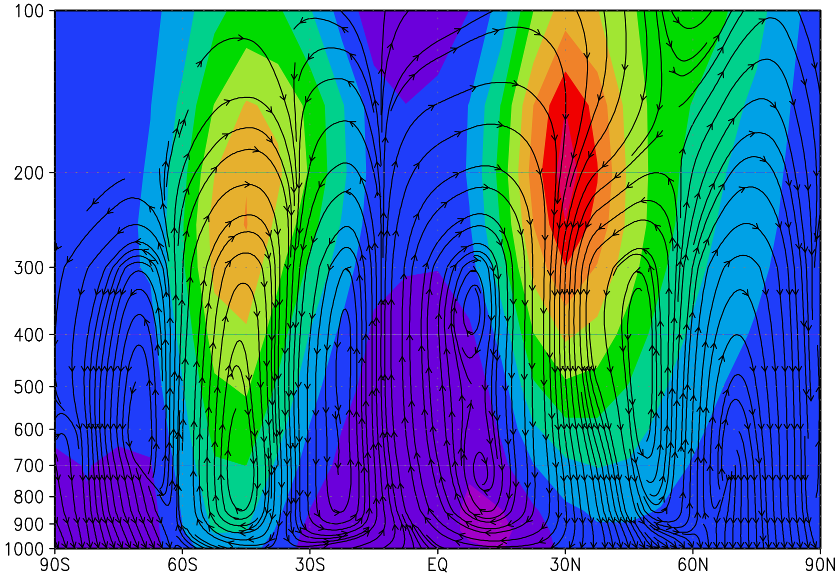
The stream function can be calculated from the $[\overline{v}]$ field. It is enough to integrate the first equation from the top of the atmosphere with $\psi = 0$ as boundary condition.

Meridional overturning circulation of the atmosphere

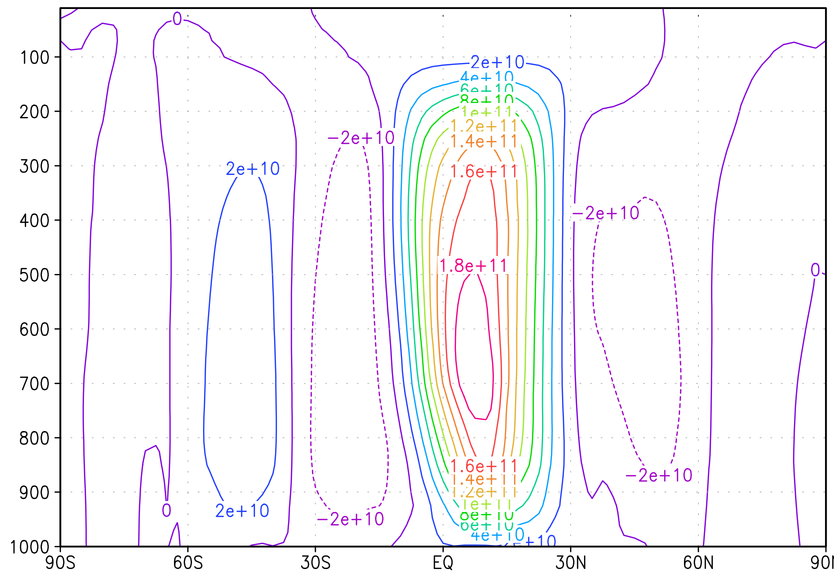


Meridional overturning circulation of the atmosphere

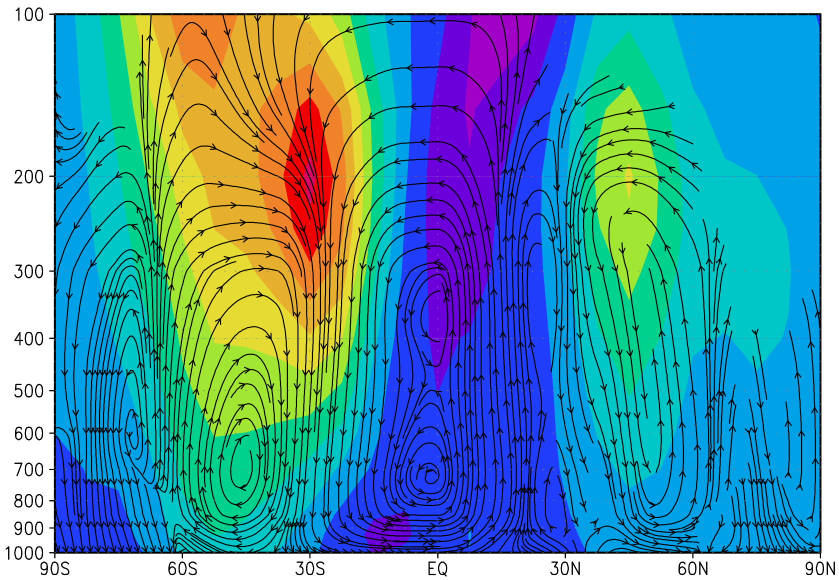
Zonal−mean merid. circul. & zonal wind, Janua



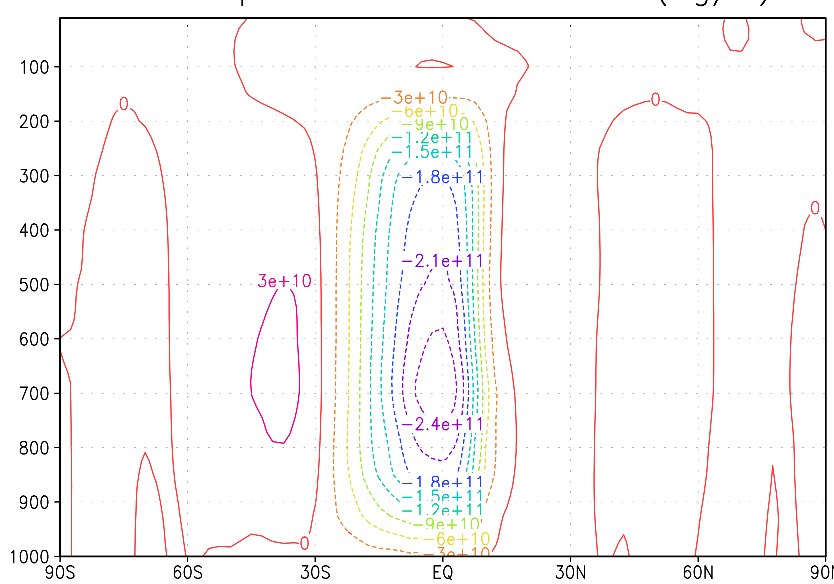
Mass transport stream function (kg/s) DJF



Zonal−mean merid. circul. & zonal wind, July



Mass transport stream function (kg/s) JJA



Thermal wind equation to establish a relationship between u and T

At this point, we need to use the geostrophic approximation and hydrostatic approximation

$$f[u] = -\frac{\partial[\Phi]}{\partial y}$$

$$\frac{\partial[\Phi]}{\partial p} = \frac{R[T]}{p}$$

to deduce the thermal wind equation

$$f\frac{\partial[u]}{\partial p} = \frac{R}{p}\frac{\partial[T]}{\partial y}$$

If we take the time derivative of the thermal wind equation, we have:

$$f\frac{\partial}{\partial p}\left(\frac{\partial[u]}{\partial t}\right) = \frac{R}{p}\frac{\partial}{\partial y}\left(\frac{\partial[T]}{\partial t}\right)$$

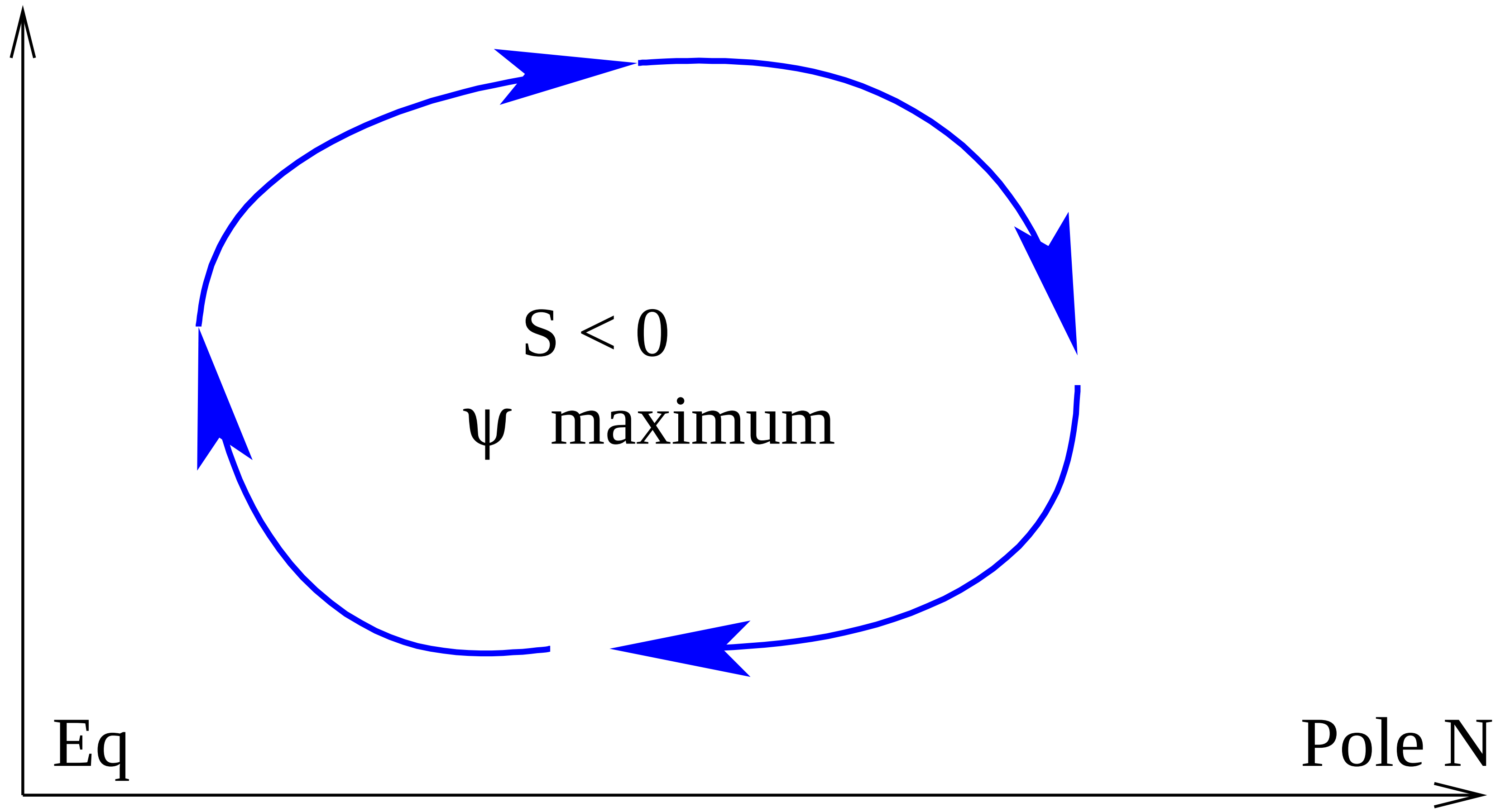
Equation governing ψ

Replace the terms $\partial[u]/\partial t$ and $\partial[T]/\partial t$ by the above equations, we obtain a diagnostic elliptique equation that governs the meridional circulation:

$$f^2\frac{\partial^2\psi}{\partial p^2} + \sigma\frac{\partial^2\psi}{\partial y^2} = f\frac{\partial^2[v^\star u^\star]}{\partial p\partial y} - \frac{R}{p}\frac{\partial^2[v^\star T^\star]}{\partial y^2} - f\frac{\partial[F_x]}{\partial p} + \frac{R}{C_p p}\frac{\partial[J]}{\partial y} = S$$

where σ is the static stability

$$\sigma \equiv \frac{RT}{C_p P} - \frac{\partial[T]}{\partial p}$$



Schematic of the meridional stream function

If the values of ψ on the boundaries are known, one can numerically solve this elliptic equation. This equation is also useful to diagnose qualitatively the mean meridional circulation. Since ψ must vanish on the boundaries, it can be represented by a double Fourier series in y and p.

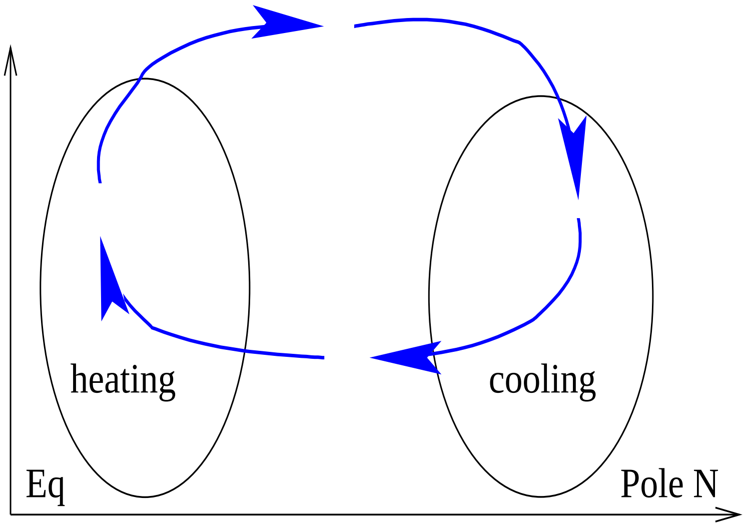
$$\psi = \sum_{m=1}^{M} \sum_{n=1}^{N} A_m B_n \sin(m\pi \frac{p}{\delta_p}) \sin(n\pi \frac{y}{\delta_y})$$

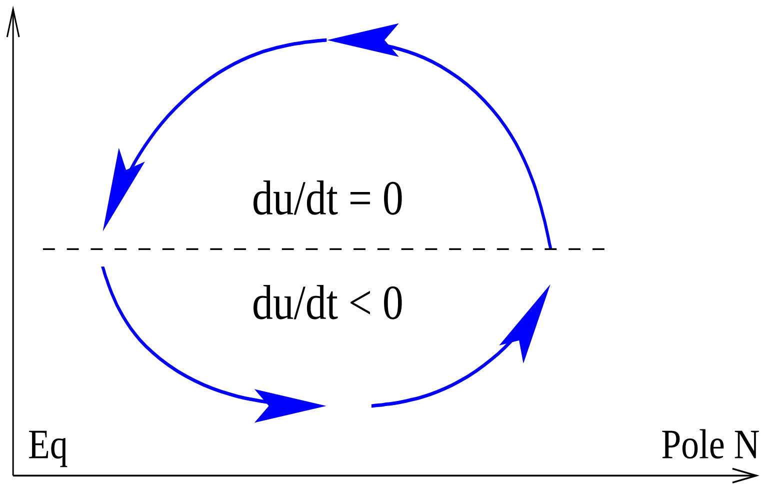
Hence, the elliptic operator on the left is approximately proportional to $-\psi$.

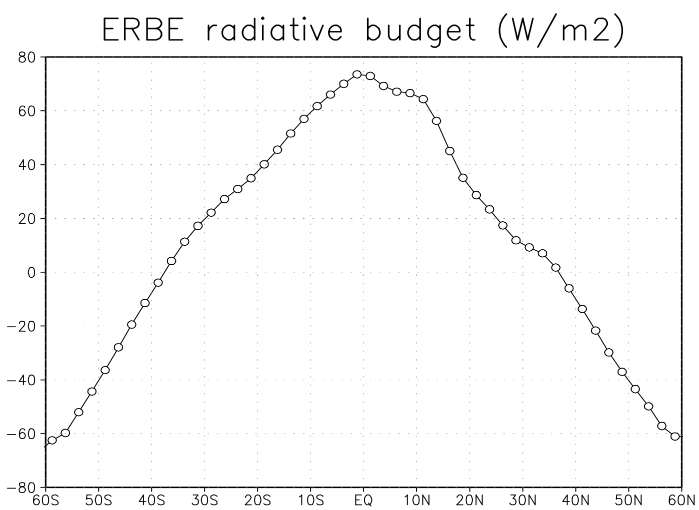
The four terms on the right, considered as sources, represent eddy momentum flux, eddy heat flux, zonal drag force and diabatic heating.

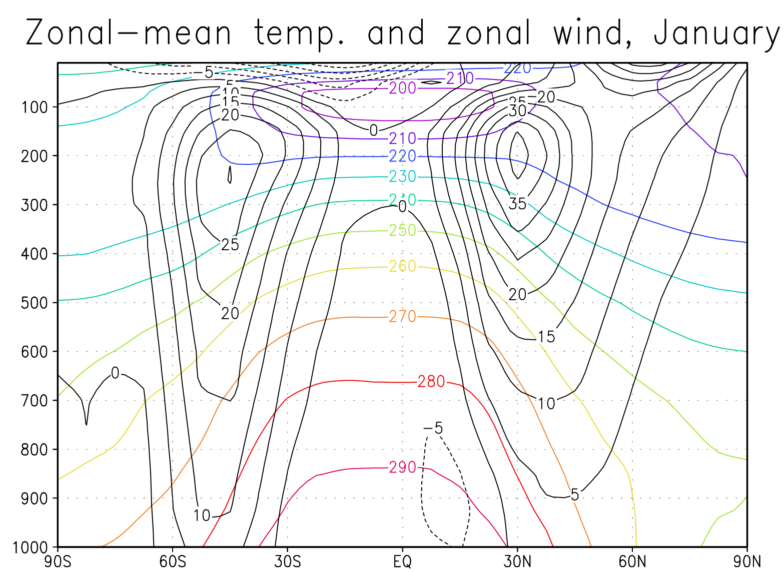
$$\psi \propto -f \frac{\partial^2 [v^\star u^\star]}{\partial p \partial y}; \frac{R}{p} \frac{\partial^2 [v^\star T^\star]}{\partial y^2}; f \frac{\partial [F_x]}{\partial p}; -\frac{R}{C_p p} \frac{\partial [J]}{\partial y}$$

$$\psi \propto -f\frac{\partial^2[v^\star u^\star]}{\partial p \partial y};\ \frac{\partial^2[v^\star T^\star]}{\partial y^2};\ f\frac{\partial [F_x]}{\partial p};\ -\frac{\partial [J]}{\partial y}$$









$$\psi \propto -f\frac{\partial^2[v^\star u^\star]}{\partial p\partial y};\ \frac{\partial^2[v^\star T^\star]}{\partial y^2};\ f\frac{\partial[F_x]}{\partial p};\ -\frac{\partial[J]}{\partial y}$$