Conservation equations for angular momentum

For a particle located by the position vector $\vec{r}$ in an inertial reference, moving with a velocity $\vec{v}$, its absolute angular momentum per unit mass is $\vec{M} = \vec{r} \times \vec{v}$. If a force $\vec{F}$ is applied to the particle, conservation of momentum requires

$$\frac{d\vec{M}}{dt} = \vec{r} \times \vec{F}$$

On the earth, the angular momentum about the axis of rotation is $M = \Omega a^2 \cos^2 \phi + ua \cos \phi$ where Ω is the angular velocity of the solid earth, u is the relative zonal velocity, a is the radius of the earth and ϕ is the latitude. The first term is the earth's angular momentum (M_e) and the second term is the relative angular momentum (M_a)

There are two significant forces that can act to change M in the atmosphere, namely pressure forces and friction:

$$\frac{dM}{dt} = a \cos \phi F - \frac{1}{\rho} \frac{\partial p}{\partial \lambda}$$

The friction force can be expressed in terms of stress tensor.

Multiplying through by ρ and invoking the continuity equation:

$$\frac{d(\rho M)}{dt} = \frac{\partial}{\partial t}(\rho M) + \nabla \cdot (\rho M \vec{v}) = -a \cos\phi \frac{\partial \tau}{\partial z} - \frac{\partial p}{\partial \lambda}$$

This equation is suitable for volume integration. We can thus obtain:

$$\frac{\partial}{\partial t} \int (\rho M) dV = T_M + T_F$$

where

$$T_M = \int_{-\pi/2}^{\pi/2} a^2 \cos\phi d\phi \int_{z_s}^{\infty} dz \sum_i (p_E^i - p_W^i)$$

$$T_F = \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} a^3 \cos^2 \phi \tau_0 d\lambda d\phi$$

Thus the total angular momentum of the atmosphere can only be changed by mountain torques or surface stresses. τ_0 is the surface wind stress, z_s is the elevation of the surface, and p_E^i and p_W^i are the pressures at the east and west sides of the ith mountain range. An alternative method to evaluate the mountain torque is from surface data alone:

$$T_M = -\int_{-\pi/2}^{\pi/2}\int_0^{2\pi} a^2 \cos\phi (p\frac{\partial z_s}{\partial \lambda}) d\lambda d\phi$$

The mountain torque is the torque about a given axis exerted on or by the earth's surface due to the force associated with a difference of pressure on two sides of a mountain. For example, if the axis is the earth's polar axis and the air pressure is higher (level for level) on the west side of a mountain than on the east, there exists a mountain torque in the vicinity tending to speed up the earth's rotation.

Balance equation for a latitudinal belt

For the relative angular momentum, since $M = \Omega a^2 \cos^2 \phi + M_a$, we obtain the equation:

$$\frac{dM_a}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial \lambda} + a\cos\phi(fv + F)$$

Now we multiply the equation by ρ and write the new equation in flux form:

$$\frac{\partial}{\partial t}(\rho M_a) + \frac{1}{a\cos\phi}\frac{\partial}{\partial \lambda}(\rho u M_a)$$
$$+\frac{1}{a\cos\phi}\frac{\partial}{\partial \phi}(\rho v\cos\phi M_a) + \frac{\partial(\rho w M_a)}{\partial z}$$
$$= -\frac{\partial p}{\partial \lambda} + a\cos\phi(\rho f v - \frac{\partial \tau}{\partial z})$$

Now we consider the balance that maintains ρM_a, the relative angular momentum of the atmosphere. We make firstly a vertical integration, applying the boundary conditions that $w = 0$ at $z = z_s, \infty$.

$$\frac{\partial}{\partial t}(\int_{z_s}^{\infty} \rho M_a dz) + \frac{1}{a\cos\phi}\frac{\partial}{\partial\lambda}(\int_{z_s}^{\infty} \rho u M_a dz)$$
$$+\frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}(\int_{z_s}^{\infty} \rho v\cos\phi M_a dz)$$
$$= -\int_{z_s}^{\infty}\frac{\partial p}{\partial\lambda}dz + a\cos\phi\int_{z_s}^{\infty}\rho f v dz + a\cos\phi\tau_0$$

Next we integrate zonally, the second terms for both sides vanish:

$$\frac{\partial}{\partial t}(\int_0^{2\pi} d\lambda\int_{z_s}^{\infty} \rho M_a dz) + \frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}(\int_0^{2\pi} d\lambda\int_{z_s}^{\infty} \rho v\cos\phi M_a dz)$$
$$= \int_{z_s}^{\infty}\sum_i (P_E^i - p_W^i) dz + a\cos\phi\int_0^{2\pi}\tau_0 d\lambda$$

We use the [ ] operator to represent the zonal average:

$$\frac{\partial}{\partial t}\left(\int_{z_s}^{\infty}\rho[M_a]dz\right)+\frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}\left(\int_{z_s}^{\infty}\rho[vM_a]\cos\phi dz\right)$$
$$=\int_{z_s}^{\infty}\sum_i(P_E^i-p_W^i)dz+a\cos\phi[\tau_0]$$

If the long-term average operator is also applied, we obtain then the angular momentum budget equation for a zonal ring of air of unit meridional width, extending from the ground to the top of the atmosphere:

$$\frac{\partial}{\partial t}\left(\int_0^{p_s}[\overline{M_a}]dp\right)=$$
$$-\frac{1}{a\cos\phi}\int_0^{p_s}\frac{\partial([\overline{vM_a}]\cos\phi)}{\partial\phi}dp$$
$$+\frac{g}{2\pi}\int_{z_s}^{\infty}\sum_i(\overline{P_E^i}-\overline{p_W^i})dz+ag\cos\phi[\overline{\tau_0}]$$

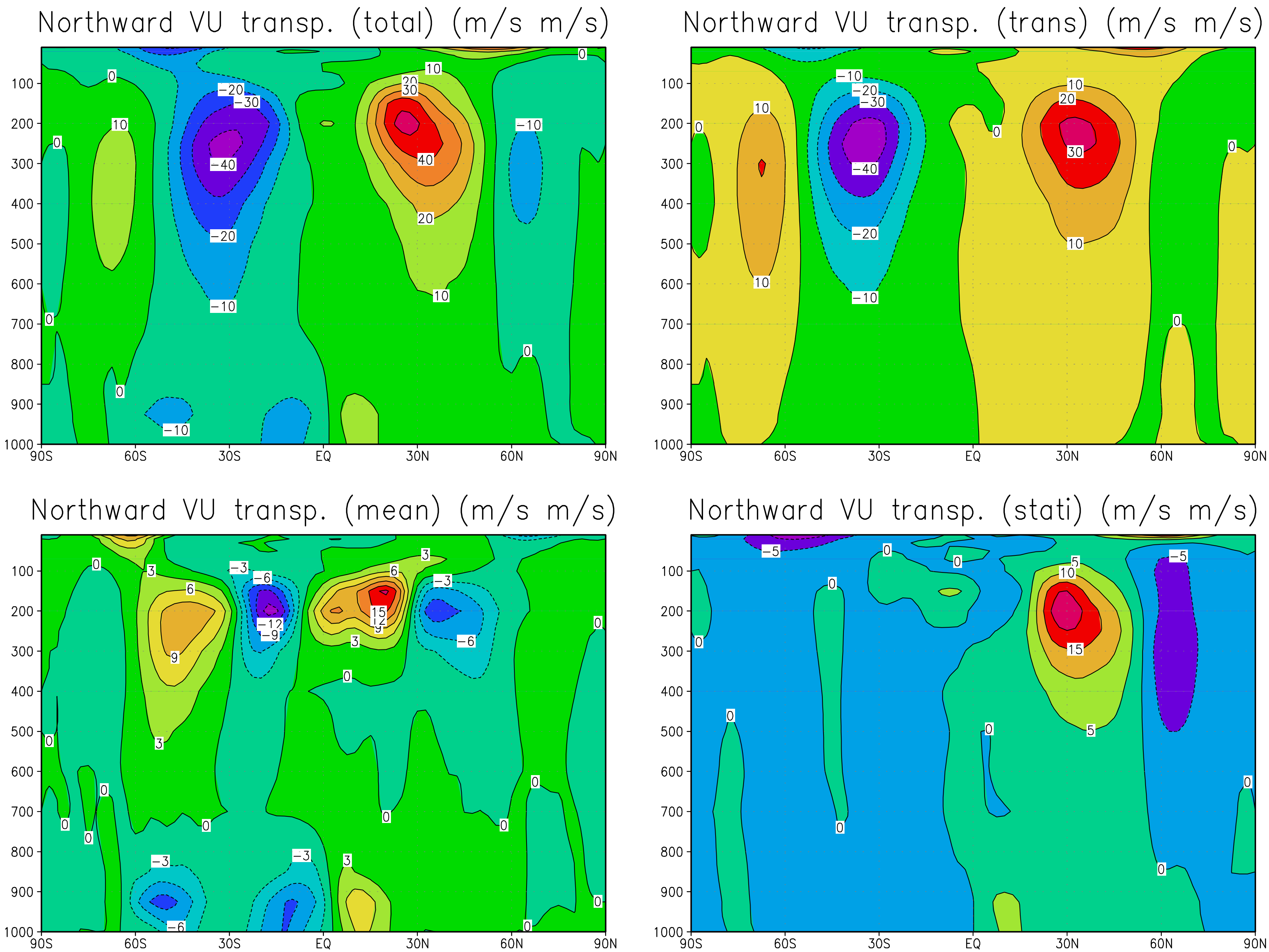
For a stationary state (long-term average), the left-side term vanishes, the divergence of relative angular momentum in a latitude belt must be balanced by mountain torques and surface stress.

The dynamical transport term can be written as:

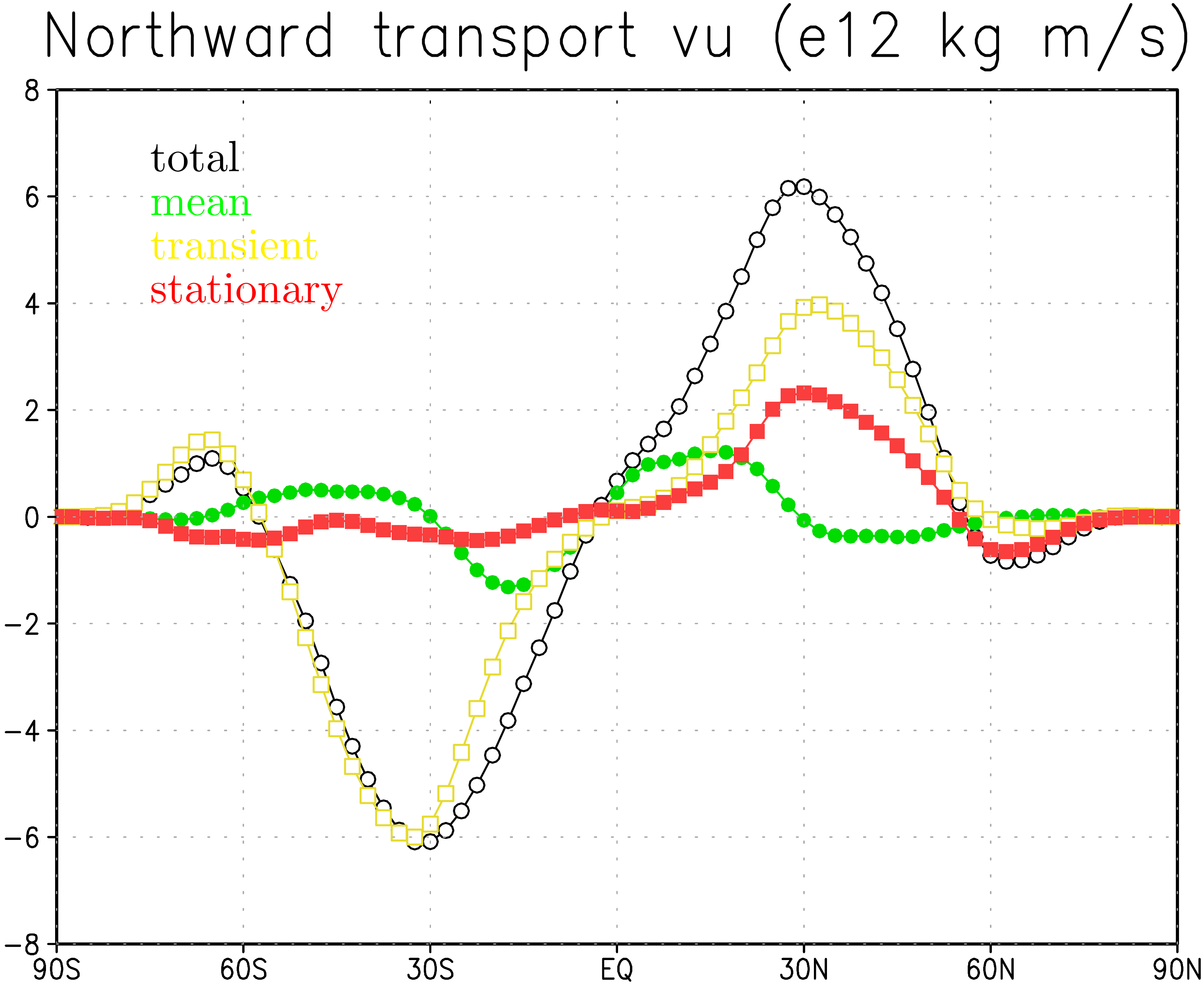
$$[\overline{vM_a}] = a\cos\phi[\overline{vu}] = a\cos\phi\{[\overline{v}][\overline{u}] + [\overline{v}^\star\overline{u}^\star] + [\overline{v'u'}]\}$$

It is thus splitted into three terms: transport by mean circulation, stationary eddies and transient eddies.

Zonally and annually averaged of meridional transport for u



Vertical integration of vu, annual average



Friction torque and mountain torque

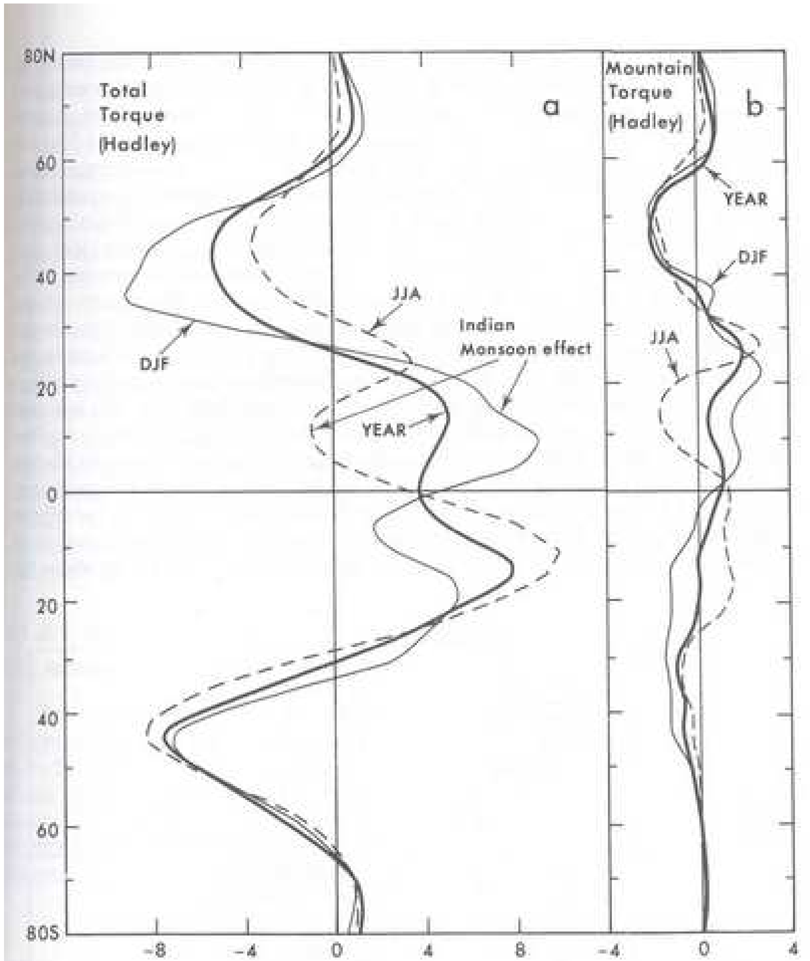


FIGURE 11.12. Meridional profiles of the mean surface torque (due to friction and mountains) (a) integrated over 5° latitude belts in Hadleys (1 Hadley = 10^{18} kg m^2 s^{-2}) and the mean mountain torque (b) after Newton (1971a) in the same units (after Oort and Peixoto, 1983).

The angular momentum exchange between the atmosphere and the surface is accomplished by the mountain torque and the frictional torques. These two terms can be estimated either directly or indirectly. The mountain torque can in principle be calculated from fields of surface pressure and surface topography. The frictional torque is difficult to have its direct estimation, since the surface stress $\tau_0 = -\rho C_D |\vec{v_s}| u_s$ is generally not resolved by the station observations. It can be however indirectly estimated as a residual term.