Derivative of a vector in a rotating system

For a whatever vector $\vec{A}$,

$$\frac{d_a\vec{A}}{dt} = \frac{d\vec{A}}{dt} + \vec{\Omega} \times \vec{A}$$

where $\vec{\Omega}$ is the angular vector of the rotating system. $d_a\vec{A}/dt$ iis the derivative of the vector $\vec{A}$ in the inertial system. and $d\vec{A}/dt$ is the derivative of the vector $\vec{A}$ in the relative system.

The general relationship can be thus expressed as:

$$\frac{d_a}{dt} = \frac{d}{dt} + \vec{\Omega}\times$$

Expression of the acceleration term in rotating coordinates

By applying the general relationship to the position vector $\vec{r}$ (defined as a vector from the Earth's centre to the considered point at the Earth's surface), we obtain the equation relating the absolute velocity and the relative velocity:

$$\frac{d_a\vec{r}}{dt} = \frac{d\vec{r}}{dt} + \vec{\Omega} \times \vec{r}$$

That is:

$$\vec{V}_a = \vec{V} + \vec{\Omega} \times \vec{r}$$

where $\vec{\Omega} \times \vec{r}$ is the associated velocity due to the rotation of the Earth.

The general derivative operator can be applied to the velocity vector itself $\vec{V}_a$:

$$\frac{d_a\vec{V}_a}{dt} = \frac{d\vec{V}_a}{dt} + \vec{\Omega} \times \vec{V}_a$$

This gives:

$$\begin{aligned}\frac{d_a\vec{V}_a}{dt} &= \frac{d}{dt}(\vec{V}+\vec{\Omega}\times\vec{r})+\vec{\Omega}\times(\vec{V}+\vec{\Omega}\times\vec{r})\\ &= \frac{d\vec{V}}{dt}+2\vec{\Omega}\times\vec{V}+\vec{\Omega}\times(\vec{\Omega}\times\vec{r})\\ &= \frac{d\vec{V}}{dt}+2\vec{\Omega}\times\vec{V}-\Omega^2\vec{R}\end{aligned}$$

where $\vec{R}$ is a vector perpendicular to the axis of rotation, with magnitude equal to the distance to the axis of rotation. The absolute acceleration $d_a\vec{V}_a/dt$ can thus be decomposed into three terms:

- relative acceleration $d\vec{V}/dt$,
- Coriolis acceleration $2\vec{\Omega}\times\vec{V}$ and
- centripetal acceleration $-\Omega^2\vec{R}$.

Equations governing the atmosphere

- Equation of motion, momentum conservation
- Continuity equation, mass conservation
- Thermodynamic energy equation, energy conservation
- Equation of state for an ideal gas

Equation of motion

For a particle of air of unit mass with ρ as the density, different forces can be identified, either "real" or "apparent".

- The pressure gradient force $\vec{F} = -\frac{1}{\rho}\nabla p$.
- The viscous force $\vec{N} = \nu\nabla^2\vec{V}$ where ν is the viscosity coefficient.
- The gravitational force $\vec{g} = -\frac{GM}{r^3}\vec{r}$ where G is the gravitational constant.
- The centrifugal force $\Omega^2\vec{R}$
- The Coriolis force $-2\vec{\Omega}\times\vec{V}$
- The gravity force $\vec{g} = -\frac{GM}{r^3}\vec{r} + \Omega^2\vec{R} = -\nabla\Phi$. In practice, $|\vec{g}|$ changes only about 1% at maximum around the constant m/s^2. The angle between $\vec{g}$ and $\vec{r}$ is always smaller than $0.2°$.

The centrifugal force and the Coriolis force are apparent ones, due to the rotation of the Earth.

We have now the dynamic equation of the atmosphere:

$$\frac{d\vec{V}}{dt} = -\frac{1}{\rho}\nabla p - 2\vec{\Omega} \times \vec{V} + \vec{g} + \vec{N}$$

Mass conservation equation

Mass divergence form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$$

Or velocity divergence form:

$$\frac{d\rho}{dt} + \rho \nabla \cdot \vec{V} = 0$$

Equation of state

The thermodynamic state of the atmosphere at any point is determined by the values of pressure, temperature, and density (or specific volume) at that point. These field variables are related to each other by the equation of state for ideal gas.

$$p = \rho R T$$

Energy conservation equation

If the heating rate of a particle of air is Q (J kg^{-1}s^{-1}), the internal energy has to change $C_v \frac{dT}{dt}$ and the particle works through compression or expansion by $p\frac{d}{dt}(\frac{1}{\rho})$. The enrgy conservation equation is thus:

$$C_v \frac{dT}{dt} + p\frac{d}{dt}(\frac{1}{\rho}) = Q$$

By using the equation of state, we can obtain:

$$p\frac{d}{dt}(1/\rho) = R\frac{dT}{dt} - \frac{1}{\rho}\frac{dp}{dt}$$

The energy conservation equation can thus be expressed in another form:

$$C_p \frac{dT}{dt} - \frac{1}{\rho}\frac{dp}{dt} = Q$$

where $C_p = C_v + R$ is the specific heat at constant pressure.

Potential temperature

For an ideal gas undergoing an adiabatic process (i.e., a reversible process in which no heat is exchanged with the surroundings, $Q = 0$) the energy conservation equation can be written in the form

$$C_p d \ln T - R d \ln p = 0$$

Integrating this expression from a state at pressure p and temperature T to a state in which the pressure is p_0 and the temperature is θ, we obtain after taking the antilogarithm

$$\theta = T \left(\frac{p_0}{p} \right)^{R/C_p}$$

This relationship is referred to as Poisson's equation, and the temperature θ thus defined is called the potential temperature.

The spherical coordinates

The three primative directions over the Earth are the longitude λ, latitude ϕ and the radial distance r from the center of the Earth. In practice, the third direction uses z, the distance from the sea level. Suppose that the radius of the Earth (a) is constant, one can obtain $r = a + z$ and $dr = dz$.

By using u, v and w to represent the three components of the velocity vector, we can have:

$$u = r \cos\phi \frac{d\lambda}{dt}$$

$$v = r \frac{d\phi}{dt}$$

$$w = \frac{dr}{dt}$$

Equation of motion in spherical coordinates

For purpose of theoretical analysis and numerical prediction, it is necessary to expand the vectorial equation of motion into its scalar components:

$$\frac{du}{dt} = \frac{uv\tan\phi}{r} - \frac{uw}{r} - \frac{1}{\rho r\cos\phi}\frac{\partial p}{\partial\lambda} + fv - \hat{f}w + N_\lambda$$

$$\frac{dv}{dt} = -\frac{u^2\tan\phi}{r} - \frac{vw}{r} - \frac{1}{\rho r}\frac{\partial p}{\partial\phi} - fu + N_\phi$$

$$\frac{dw}{dt} = \frac{u^2+v^2}{r} - \frac{1}{\rho}\frac{\partial p}{\partial r} - g + \hat{f}u + N_z$$

The terms proportional to $1/r$ are called the curvature terms, owing to the curvature of the earth. $f = 2\Omega\sin\phi$ is the Coriolis parameter and $\hat{f} = 2\Omega\cos\phi$. Considering $r = a + z$ with $a \gg z$, one can replace r by a and dr by dz in the above equations.

Scale analysis of the equations of motion

For synoptic-scale circulations, one can have:

$$\frac{du}{dt} = \frac{uv\tan\phi}{a} - \frac{uw}{a} - \frac{1}{\rho a\cos\phi}\frac{\partial p}{\partial\lambda} + fv - \hat{f}w + N_\lambda$$

$\frac{du}{dt}$		$\frac{uv\tan\phi}{a}$	$-\frac{uw}{a}$	$-\frac{1}{\rho a\cos\phi}\frac{\partial p}{\partial\lambda}$	$+fv$	$-\hat{f}w$	$+N_\lambda$
10^{-4}		10^{-5}	10^{-8}	10^{-3}	10^{-3}	10^{-6}	10^{-5}

$$\frac{dv}{dt} = -\frac{u^2\tan\phi}{a} - \frac{vw}{a} - \frac{1}{\rho a}\frac{\partial p}{\partial\phi} - fu + N_\phi$$

$\frac{dv}{dt}$		$-\frac{u^2\tan\phi}{a}$	$-\frac{vw}{a}$	$-\frac{1}{\rho a}\frac{\partial p}{\partial\phi}$	$-fu$	$+N_\phi$
10^{-4}		10^{-5}	10^{-8}	10^{-3}	10^{-3}	10^{-5}

$$\frac{dw}{dt} = \frac{u^2v^2}{a} - \frac{1}{\rho}\frac{\partial p}{\partial z} - g + \hat{f}u + N_z$$

$\frac{dw}{dt}$		$\frac{u^2v^2}{a}$	$-\frac{1}{\rho}\frac{\partial p}{\partial z}$	$-g$	$+\hat{f}u$	$+N_z$
10^{-7}		10^{-5}	10	10	10^{-3}	10^{-7}

Hydrostatic Approximation:

$$\frac{\partial p}{\partial z} = -\rho g$$

Geostropic Approximation: the pressure gradient force is approximatively in equilibrium with the Coriolis force:

$$f v_g = \frac{1}{\rho a \cos\phi} \frac{\partial p}{\partial \lambda}$$

$$f u_g = -\frac{1}{\rho a} \frac{\partial p}{\partial \phi}$$

Or in vector form:

$$\vec{v}_g = \frac{1}{\rho f} \vec{k} \times \nabla p$$

One can see that the geostrophic balance is a diagnostic expression that gives the approximative relationship between the pressure field and horizontal velocity in large-scale extratropical systems. The geostrophic approximation is verified when the wind acceleration is much smaller than the Coriolis force.

Thermal wind

This is an important notion to understand the general circulation of the atmosphere. The thermal wind ($\vec{V}_T$) is defined as the vertical variation of the geostrophic wind:

$$\vec{V}_T = \frac{\partial \vec{V}_g}{\partial z} = \frac{R}{f}\vec{k} \times \nabla T$$

A horizontal temperature gradient can thus induce a wind vector that is parallel to the iso-temperature lines.

Primitive Equation

The geostrophic equation is a diagnostic relation, since it does contain any terms of temporal derivative. One needs to take other terms with smaller values to obtain a **prognostic equation**:

$$\frac{du}{dt} = fv - \frac{1}{\rho a \cos\phi}\frac{\partial p}{\partial \lambda} + N_\lambda$$

$$\frac{dv}{dt} = -fu - \frac{1}{\rho a}\frac{\partial p}{\partial \phi} + N_\phi$$

For phenomena with a spatial extension comparable to the radius of the Earth, such as stationary planetary waves, the Hadley cell, one needs also to take into consideration the terms in relation with the curvature of the Earth.

$$\frac{du}{dt} = \frac{uv}{a}\tan\phi + fv - \frac{1}{\rho a \cos\phi}\frac{\partial p}{\partial \lambda} + N_\lambda$$

$$\frac{dv}{dt} = \frac{u^2}{a}\tan\phi - fu - \frac{1}{\rho a}\frac{\partial p}{\partial \phi} + N_\phi$$

Vorticity and potential vorticity

Vorticity is a vector field defined as the curl of 3-dimensional velocity field $\nabla \times \vec{V}_{3d}$. It is the microscopic measure of rotation in a fluid. The absolute vorticity is the curl of the absolute velocity, while the relative vorticity is the curl of the relative velocity. Although the vorticity is a vector field, we are in general concerned only with its vertical component, since the motions are mainly horizontal for large-scale dynamics of the atmosphere: $\vec{k}\cdot(\nabla\times\vec{V}_{3d})$, that is, $\zeta = \nabla\times\vec{V} = \partial v/\partial x - \partial u/\partial y$. Vorticity equation can be deduced from equation of motion.

The potential vorticity is defined as the projection of the vorticity vector $\nabla\times\vec{V}_{3d}$ onto the direction indicated by the potential temperature gradient $\nabla\theta$:

$$P = \frac{1}{\rho}(\nabla\theta)\cdot(\nabla\times\vec{V}_{3d}).$$

By using the equation of motion, the continuity equation and the thermodynamical equation, one can deduce the equation governing the potential vorticity. For an adiabatic motion without friction, a particle of air conserves its potential vorticity.

Equation of vorticity

In Cartesian coordinates:

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-fv=-\frac{1}{\rho}\frac{\partial p}{\partial x}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+fu=-\frac{1}{\rho}\frac{\partial p}{\partial y}\right)$$

One can thus obtain:

$$\frac{d}{dt}(\zeta+f)=-(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)+\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

Three terms contribute to the rate of change for absolute vorticity: the divergence term, the tilting or twisting term, and the solenoidal term. For synoptic-scale atmospheric circulation, the divergence term is dominant: a horizontal convergence generates vorticity and a horizontal divergence destroys vorticity.

Furthermore, if the motion is non-divergent, the absolute vorticity is conserved.

$$\frac{d}{dt}(\zeta + f) = 0.$$

That is (with $\partial f/\partial x = 0$ and $\beta = \partial f/\partial y$):

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\zeta + \beta v = 0.$$

This equation, studied by C. G. Rossby at the end of 1930, also known as Rossby Equation, is very useful for meteorological phenomena. The first numerical weather prediction was realized by J. Charney and his group at the beginning of 1950 through the Rossby equation.